very large the value of the fraction is *very very small*, and when the denominator is *infinitely large* or ∞, its value is *infinitely small* or

$$\frac{m}{\infty}=0. \quad \ldots \quad \ldots \quad \ldots \quad \text{(ii)}$$

Cor. When $m=1, \frac{1}{\infty}=0.$

(iii) If the numerator of a fraction is constant (say m), its value increases as its denominator decreases, *e.g.*, $\frac{m}{\frac{1}{3}} > \frac{m}{\frac{1}{2}}$ and $\frac{m}{\frac{1}{4}} > \frac{m}{\frac{1}{3}}$ ...and when the denominator is *very very small*, *the* value of the fraction is *very very large*, and when the denominator is *infinitely small* or 0, its value is *infinitely large*, or

$$\frac{m}{0}=\infty. \quad \ldots \quad \ldots \quad \ldots \quad \text{(iii)}$$

Cor. When $m=1, \frac{1}{0}=\infty.$

(iv) $\frac{x^2-1}{x-1}=\frac{1-1}{1-1}=\frac{0}{0}$ when $x=1$ (By substitution).

But $\frac{x^2-1}{x-1}\equiv x+1=1+1=2$ when $x=1$ (By identity).

$$\therefore \quad \frac{0}{0}=2. \quad \ldots \quad \ldots \quad \ldots \quad \text{(i)}$$

Again, $\frac{x^3-1}{x-1}=\frac{1-1}{1-1}=\frac{0}{0}$ when $x=1$ (By substitution).

But $\frac{x^3-1}{x-1}\equiv x^2+x+1=1+1+1=3$ when $x=1$

(By identity).

$$\therefore \quad \frac{0}{0}=3 \quad \ldots \quad \ldots \quad \ldots \quad \text{(ii)}$$

Again, $\frac{x^2-1}{x^3-1}=\frac{1-1}{1-1}=\frac{0}{0}$ when $x=1$ (By substitution).

But $\frac{x^2-1}{x^3-1}\equiv\frac{x+1}{x^2+x+1}=\frac{1+1}{1+1+1}=\frac{2}{3}$ when $x=1$

(By identity).

$$\therefore \quad \frac{0}{0}=\frac{2}{3}. \quad \ldots \quad \ldots \quad \ldots \quad \text{(iii)}$$

Again, $\frac{x^3-1}{x^4-1} = \frac{1-1}{1-1} = \frac{0}{0}$ when $x=1$ (By substitution).

But $\frac{x^3-1}{x^4-1} \equiv \frac{x^2+x+1}{x^3+x^2+x+1} = \frac{1+1+1}{1+1+1+1} = \frac{3}{4}$ when $x=1$.

(By identity).

$$\therefore \quad \frac{0}{0} = \frac{3}{4} \quad \ldots \quad \ldots \quad \ldots \quad \text{(iv)}$$

Thus, $\frac{0}{0}$ has several *different* values. By taking suitable identities and proceeding as above, we can prove that $\frac{0}{0}$ can have *any* value or is *indeterminate*.

Example 6. Find the value of $\frac{x^2+ax-2a^2}{x^2+4ax-5a^2}$ when $x=a$.

If we substitute a for x in the given expression, it is reduced to the indeterminate form $\frac{0}{0}$.

In order to avoid the indeterminate form, we proceed as follows:

$$\text{The expression} = \frac{(x+2a)(x-a)}{(x+5a)(x-a)} \equiv \frac{x+2a}{x+5a}$$

$$= \frac{a+2a}{a+5a}$$

$$= \frac{3a}{6a} = \tfrac{1}{2}.$$

Find the value of:

*33. $\frac{2x^2+5x-7}{3x^2-8x+5}$ when $x=1$.

*34. $\frac{x^2+3x-40}{x^2+4x-32}$ when $x=-8$.

*35. $\frac{x^3+4x^2-8x+3}{2x^3-5x^2+2x+1}$ when $x=1$.

*36. $\frac{2x^3-7x^2a+6a^2x-a^3}{3x^4-3x^3a+4a^3x-4a^4}$ when $x=a$.

*37. $\frac{3x^3-27ax^2+78a^2x-72a^3}{2x^3+10ax^2-4a^2x-48a^3}$ when $x=2a$.

*38. $\frac{x^3-5ax^2+3a^2x+9a^3}{x^3-7ax^2+15a^2x-9a^3}$ when $x=3a$.

CHAPTER XIII

PROPORTION

1. If $\frac{a}{b}=\frac{c}{d}$, the numbers a, b, c, d are said to be **in proportion** (or **proportionals**) and it is read as 'a to b equals c to d' or 'a is to b as c is to d'.

Of this proportion a, d are called the **extremes,** b, c the **means** and d the **fourth proportional** to a, b and c.

Two quantities are said to be *proportional* to two others when the ratio of the first two is equal to the ratio of the second two, and two quantities are said to be *inversely proportional* to two others when the *inverse* ratio of the first two is equal to the ratio of the other two.

If a, b, c, d, e...be such that $a : b=b : c=c : d=d : e=$&c., a, b, c, d, e are said to be in **continued proportion.**

When a, b, c, are in continued proportion, b is said to be the **mean proportional** between a and c and c is said to be the **third proportional** to a and b.

2. If $a : b=c : d$, then $ad=bc$.

Since $\frac{a}{b}=\frac{c}{d}$, multiplying both sides by bd, we get

$$ad=bc.$$

Thus, if *four quantities are proportional, the product of the extremes is equal to the product of the means.*

Conversely, if $ad=bc$, we can easily prove by dividing both sides by bd that $\frac{a}{b}=\frac{c}{d}$, or $a : b=c : d$.

Cor. If $a : b=b : c$, then $ac=b^2$, *i.e., if three quantities are in continued proportion, the product of the extremes is equal to the square of the means.*

3. If $a:b=b:c$, then $a:c=a^2:b^2$.

For $\frac{a}{b}=\frac{b}{c}$

$\therefore \quad \frac{a}{b}\times\frac{b}{c}=\frac{a}{b}\times\frac{a}{b}$

$\therefore \quad \frac{a}{c}=\frac{a^2}{b^2}$, or $a:c=a^2:b^2$.

Thus, if *three quantities are in continued proportion, the first is to the third in the duplicate ratio of the first to the second.*

4. If $a:b=b:c=c:d$, then $a:d=a^3:b^3$.

For $\frac{a}{b}=\frac{b}{c}=\frac{c}{d}$

$\therefore \quad \frac{a}{b}\times\frac{b}{c}\times\frac{c}{d}=\frac{a}{b}\times\frac{a}{b}\times\frac{a}{b}$

$\therefore \quad \frac{a}{d}=\frac{a^3}{b^3}$, or $a:d=a^3:b^3$.

Thus, if *four quantities are in continued proportion, the first is to the fourth in the triplicate ratio of the first to the second.*

Example 1. Find the fourth proportional to 2, 3, 4.

Let x be the fourth proportional required.

Then $\frac{2}{3}=\frac{4}{x}$, which gives $x=6$

$\therefore$ the 4th proportional is 6.

Example 2. Find the mean proportional to 9, 25.

Let x be the mean proportional required.

Then $\frac{9}{x}=\frac{x}{25}$, $\quad\therefore\quad x^2=9\times25$, $\quad\therefore\quad x=15$.

$\therefore$ the mean proportional is 15.

Example 3. Find the third proportional to 2, 8.

Let x be the third proportional required.

Then $\frac{2}{8}=\frac{8}{x}$, $\quad\therefore\quad 2x=64$, $\quad\therefore\quad x=32$.

$\therefore$ the 3rd proportional is 32.

EXERCISE 60.

Find the fourth proportional to:

1. 8, 12, 14. **2.** 9, 12, 15. **3.** ·0024, 2·4, ·04. **4.** $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$.

5. $3a$, $6a^2$, $2ab^2$.

Find the mean proportional to:

6. 9, 16. **7.** 49, 25. **8.** 9, 64. **9.** 121, 144. **10.** $4\frac{1}{2}$, 2.

11. x^2, y^2. **12.** $(a+b)(a-b)^3$ and $(a+b)^3(a-b)$.

Find the third proportional to:

13. 18, 12. **14.** 20, 16. **15.** 15, 6. **16.** x^2-y^2, $(x+y)^2$.

17. If $1+x$, $3+x$, $6+x$ are in continued proportion, find x.

Example 4. What must be added to each of the numbers 7, 11 and 19 so that the resulting numbers may be in continued proportion?

Let x be the required number.

Then $\dfrac{7+x}{11+x}=\dfrac{11+x}{19+x}$.

Multiplying cross-wise, we have

$$(7+x)(19+x)=(11+x)^2$$

$$\therefore \quad 133+26x+x^2=121+22x+x^2$$

$$\therefore \quad 4x=-12, \text{ or } x=-3.$$

$\therefore$ The number to be added is -3.

18. What must be added to each of the numbers 3, 7, 13 and 22 so that the results may be in proportion?

19. What must be subtracted from each of the numbers 27, 41, 30 and 46 so that the results may be in proportion?

20. Shew that the fourth proportional to $x^2+7x+12$, $5(x+3)$ and $6(x+4)$ is independent of x.

21. Find two numbers such that their mean proportional is 12 and the third proportional is 324.

22. The mean proportional between 45 and a certain number is three times the mean proportional between 5 and 22. Find the number.

23. If y is the mean proportional between x and z, shew that $xy+yz$ is the mean proportional between x^2+y^2 and y^2+z^2.

24. Two numbers consisting of the same two digits are in the ratio of 4 : 7. Find the numbers.

25. A certain kind of brass is made up of copper, zinc, lead and tin. The ratio of copper to zinc is 1 : 2, that of zinc to lead is 3 : 4 and that of lead to tin is 4 : 5. Find the quantity of each metal in 540 lbs. of brass.

***Example 5.** A cask is filled with wine and water in the ratio of 4 : 3. If 12 gallons of the mixture be drawn off and the cask be filled with water, their ratio becomes 3 : 4. How many gallons can the cask hold?

Let x be the required number of gallons.

When 12 gallons are drawn off, the number of gallons in it $=x-12$. Out of every 7 gallons of mixture, 4 are wine and 3 are water, or $\frac{4}{7}$ of the mixture is wine and $\frac{3}{7}$ is water.

Hence $\frac{4}{7}(x-12)$ gallons of wine are left and $\frac{3}{7}(x-12)$ gallons of water.

When the cask is again filled up with water, the quantity of water $=\frac{3}{7}(x-12)+12$.

By the question, we have

$$\frac{\frac{4}{7}(x-12)}{\frac{3}{7}(x-12)+12}=\frac{3}{4}.$$

Multiplying cross-wise, we have

$$16(x-12)=9(x-12)+252$$

$$\therefore \quad 7x=336$$

$$\therefore \quad x=48.$$

$\therefore$ the cask can hold 48 gallons.

***26.** Two vessels contain a mixture of wine and water in the ratios of 8 : 5 and 4 : 1. In what ratio must liquids be drawn from each to give a mixture of wine and water in the ratio of 3 : 1?

*27 Two vessels contain mixtures of wine and water in the ratios of 7 : 3 and 3 : 2. In what ratio must liquids be drawn from each to give a mixture of wine and water in the ratio of 9 : 5?

5. **Propositions.** When $a : b = c : d$, the following relations always hold good. The student is advised not only to reproduce them, but to learn each by heart.

I. If $a : b = c : d$, then $b : a = d : c$.

For if $\frac{a}{b} = \frac{c}{d}$, then $1 \div \frac{a}{b} = 1 \div \frac{c}{d}$

$$\text{or} \quad \frac{b}{a} = \frac{d}{c}$$

$$\text{or} \quad b : a = d : c.$$

This proposition is known as **Invertendo.**

II. If $a : b = c : d$, then $a : c = b : d$.

For if $\frac{a}{b} = \frac{c}{d}$, then $ad = bc$.

Dividing both sides by cd, we get

$$\frac{a}{c} = \frac{b}{d}$$

$$\text{or} \quad a : c = b : d.$$

This proposition is known as **Alternando** and is frequently used in the solution of fractional equations.

III. If $a : b = c : d$, then $(a + b) : b = (c + d) : d$.

For if $\frac{a}{b} = \frac{c}{d}$, then $\frac{a}{b} + 1 = \frac{c}{d} + 1$

$$\therefore \quad \frac{a+b}{b} = \frac{c+d}{d}$$

$$\text{or} \quad (a + b) : b = (c + d) : d.$$

This proposition is known as **Componendo.**

IV. If $a : b = c : d$, then $(a - b) : b = (c - d) : d$.

For if $\frac{a}{b} = \frac{c}{d}$, then $\frac{a}{b} - 1 = \frac{c}{d} - 1$.

$$\therefore \quad \frac{a-b}{b}=\frac{c-d}{d}$$

or $(a-b):b=(c-d):d.$

This proposition is known as **Dividendo.**

V. If $a:b=c:d$, then $(a+b):(a-b)=(c+d):(c-d)$.

By III, we have $\frac{a+b}{b}=\frac{c+d}{d}$.

By IV, we have $\frac{a-b}{b}=\frac{c-d}{d}$.

By division, we have $\frac{a+b}{a-b}=\frac{c+d}{c-d}$.

Or $(a+b):(a-b)=(c+d):(c-d)$.

This proposition is known as **Componendo and Dividendo** and is frequently used in the solution of fractional equations.

Example 1. If $\frac{4a+5b}{4a-5b}=\frac{4c+5d}{4c-5d}$, then $\frac{a}{b}=\frac{c}{d}$.

By componendo and dividendo, we have

$$\frac{(4a+5b)+(4a-5b)}{(4a+5b)-(4a-5b)}=\frac{(4c+5d)+(4c-5d)}{(4c+5d)-(4c-5d)}$$

$$\therefore \quad \frac{8a}{10b}=\frac{8c}{10d}.$$

Multiplying both sides by $\frac{10}{8}$, we have

$$\frac{a}{b}=\frac{c}{d}$$

or $a:b=c:d.$

Example 2. If $a:b=c:d$, then $\frac{2a+3b}{2a-3b}=\frac{2c+3d}{2c-3d}$.

First Method.

Let $\frac{a}{b}=\frac{c}{d}=k$; then $a=bk$ and $c=dk$.

$$\frac{2a+3b}{2a-3b}=\frac{2bk+3b}{2bk-3b}=\frac{b(2k+3)}{b(2k-3)}=\frac{2k+3}{2k-3}$$

and $$\frac{2c+3d}{2c-3d}=\frac{2dk+3d}{2dk-3d}=\frac{d(2k+3)}{d(2k-3)}=\frac{2k+3}{2k-3}$$

$$\therefore \quad \frac{2a+3b}{2a-3b}=\frac{2c+3d}{2c-3d}.$$

Second Method.

Since $\frac{a}{b}=\frac{c}{d}$, $\therefore$ $\frac{a}{c}=\frac{b}{d}$ (Alternando).

or $$\frac{2a}{2c}=\frac{3b}{3d}$$

$\therefore$ $\frac{2a}{3b}=\frac{2c}{3d}$ (Alternando).

$\therefore$ $\frac{2a+3b}{2a-3b}=\frac{2c+3d}{2c-3d}$ (Componendo and dividendo.)

[The second method is more elegant, but the first is surer.]

Example 3. If $3a+4b+3c+4d : 2a+3b+2c+3d$
$=3a+4b-6c-8d : 2a+3b-4c-6d$,

then $a : b=c : d$.

Let $$\frac{3a+4b+3c+4d}{2a+3b+2c+3d}=\frac{3a+4b-6c-8d}{2a+3b-4c-6d}=k.$$

Then $3a+4b+3c+4d=(2a+3b+2c+3d)k$ ... (i)

and $3a+4b-6c-8d=(2a+3b-4c-6d)k$ ... (ii)

From (i) and (ii), we have

$a(3-2k)+b(4-3k)+c(3-2k)+d(4-3k)=0$ ... (iii)

$a(3-2k)+b(4-3k)-2c(3-2k)-2d(4-3k)=0$ ... (iv)

From (iii) and (iv), by factors and transposition, we have

$(a+c)(3-2k)=(b+d)(3k-4)$ ... ... (v)

$(a-2c)(3-2k)=(b-2d)(3k-4)$ ... ... (vi)

From (v) and (vi), we have

$\frac{a+c}{b+d}=\frac{3k-4}{3-2k}$ ... ... (vii)

$\frac{a-2c}{b-2d}=\frac{3k-4}{3-2k}$ ... ... (viii)

From (vii) and (viii), we have

$$\frac{a+c}{b+d}=\frac{a-2c}{b-2d}.$$

Multiplying cross-wise, we have

$$ab-2ad+bc-2cd=ab+ad-2bc-2cd$$

$\therefore$ $\quad -3ad=-3bc$

or $\quad ad=bc$

or $\quad \frac{a}{b}=\frac{c}{d}$

or $\quad a:b=c:d.$

EXERCISE 61.

If $a:b=c:d$, prove that

1. $a+3c:b+3d=3a+c:3b+d.$
2. $3a-2c:3b-2d=5a-4c:5b-4d.$
3. $ma+nc:mb+nd=pa+qc:pb+qd.$
4. $ma-nc:mb-nd=pa-qc:pb-qd.$
5. $a+b:ma-nb=c+d:mc-nd.$
6. $a^2+b^2:a^2-b^2=c^2+d^2:c^2-d^2.$
7. $\frac{ma+nb}{mc+nd}=\frac{b^2c}{d^2a}.$
8. $pa^2+qc^2:pb^2+qd^2=ma^2-nc^2:mb^2-nd^2.$
9. $a^2+ab+b^2:a^2-ab+b^2=c^2+cd+d^2:c^2-cd+d^2.$
10. $a^3+c^3:b^3+d^3=\left(\frac{a+c}{b+d}\right)^3.$
11. $\frac{la^2+mab+nb^2}{pa^2+qab+rb^2}=\frac{lc^2+mcd+nd^2}{pc^2+qcd+rd^2}.$
12. $\frac{(a-b)(a-c)}{a}=(a+d)-(b+c).$

[*Hint.* Simplify the left-hand side and substitute ad for bc.]

13. $4(a+b)(c+d)=bd\left\{\frac{a+b}{b}+\frac{c+d}{d}\right\}^2.$

[*Hint.* Apply componendo, then $\frac{a+b}{b}+\frac{c+d}{d}=\frac{2(a+b)}{b}=\frac{2(c+d)}{d}$;

hence $\left(\frac{a+b}{b}+\frac{c+d}{d}\right)^2=\frac{2(a+b)}{b}\times\frac{2(c+d)}{d}.$]

14. $(5a+6c)(5b-6d)=(5a-6c)(5b+6d)$.

15. If $\dfrac{2a+3b}{2a-3b}=\dfrac{2c+3d}{2c-3d}$, then $a:b=c:d$.

16. If $\dfrac{a^2+c^2}{b^2+d^2}=\dfrac{ac}{bd}$, then $a:b=c:d$.

17. If $\left(\dfrac{a-c}{b-d}\right)^2=\dfrac{a^2}{b^2}$, then $a:b=c:d$

18. If $\left(\dfrac{a+c}{b+d}\right)^3=\dfrac{a(a+c)^2}{b(b+d)^2}$, then $a:b=c:d$.

19. If $\dfrac{ac+bd}{ac-bd}=\dfrac{a^2+b^2}{a^2-b^2}$, then $a:b=c:d$.

20. If $(a+b+c+d)(a-b-c+d)=(a-b+c-d)(a+b-c-d)$, then $a:b=c:d$.

21. If $(2a+3b+5c+4d)(2a-3b-5c+4d)=(2a+3b-5c-4d)(2a-3b+5c-4d)$, then $ad:bc=15:8$.

22. If $\dfrac{3a+b-9c-3d}{a+2b-3c-6d}=\dfrac{3a+b+6c+2d}{a+2b+2c+4d}$, then $a:b=c:d$.

23. If $(a^2+b^2)(x^2+y^2)=(ax+by)^2$, then $a:x=b:y$.

24. If $2b=a+c$ and $\dfrac{2}{c}=\dfrac{1}{b}+\dfrac{1}{d}$, shew that $a:b=c:d$.

Example 4. If $x=\dfrac{2ab}{a+b}$, find the value of $\dfrac{x+a}{x-a}+\dfrac{x+b}{x-b}$.

Since $x=\dfrac{2ab}{a+b}$, $\quad\therefore\quad \dfrac{x}{a}=\dfrac{2b}{a+b}$ ... ... (i)

and $\dfrac{x}{b}=\dfrac{2a}{a+b}$ ... ... (ii)

Applying componendo and dividendo in (i) and (ii), we get $\dfrac{x+a}{x-a}=\dfrac{a+3b}{-a+b}$ ... ... (iii)

and $\dfrac{x+b}{x-b}=\dfrac{3a+b}{a-b}$ ... . (iv)

Adding (iii) and (iv), we get

$$\frac{x+a}{x-a}+\frac{x+b}{x-b}=\frac{a+3b}{-a+b}+\frac{3a+b}{a-b}$$

$$=\frac{-a-3b+3a+b}{a-b}$$

$$=\frac{2(a-b)}{a-b}=2.$$

25. If $x=\frac{4ab}{a+b}$, find the value of $\frac{x+2a}{x-2a}+\frac{x+2b}{x-2b}$.

26. If $x=\frac{6ab}{a+b}$, find the value of $\frac{x+3a}{x-3a}+\frac{x+3b}{x-3b}$.

Example 5. If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$, then $\frac{a^3+c^3}{b^3+d^3}=\frac{e^3}{f^3}$.

Let $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=k$, then $a=bk$, $c=dk$ and $e=fk$.

$\therefore \quad \frac{a^3+c^3}{b^3+d^3}=\frac{b^3k^3+d^3k^3}{b^3+d^3}=\frac{k^3(b^3+d^3)}{b^3+d^3}=k^3$.

But $\frac{e}{f}=k \qquad \therefore \quad \frac{e^3}{f^3}=k^3$

$\frac{a^3+c^3}{b^3+d^3}=\frac{e^3}{f^3}$, for each is equal to the same quantity.

If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$, prove that

27. $\frac{a+c}{b+d}=\frac{c-e}{d-f}$.

28. $\frac{b-3f}{a-3e}=\frac{d-b}{c-a}$.

29. $\frac{c^3+e^3}{d^3+f^3}=\frac{a^3-a^2c}{b^3-b^2d}$.

30. $\frac{bd-f^2}{b^2+d^2}=\frac{ac-e^2}{a^2+c^2}$.

31. $\frac{a^3+c^3+e^3}{b^3+d^3+f^3}=\frac{ace}{bdf}$.

Example 6. If $\frac{x^2+y^2}{xy}=\frac{p^2+q^2}{pq}$, then $\frac{x+y}{x-y}=\frac{p+q}{p-q}$.

Since $\frac{x^2+y^2}{xy}=\frac{p^2+q^2}{pq}$,

$\therefore \quad \frac{x^2+y^2}{2xy}=\frac{p^2+q^2}{2pq}$ [Dividing by 2.]

$\therefore \quad \frac{x^2+y^2+2xy}{x^2+y^2-2xy}=\frac{p^2+q^2+2pq}{p^2+q^2-2pq}$ [Componendo and dividendo.]

$\therefore \quad \frac{(x+y)^2}{(x-y)^2}=\frac{(p+q)^2}{(p-q)^2}$

$\frac{x+y}{x-y}=\frac{p+q}{p-q}$. [Sq. root of both sides.]

32. If $\frac{x^2+y^2}{p^2+q^2}=\frac{xy}{pq}$, then $\frac{(x^2+y^2)^2}{(x^2-y^2)^2}=\frac{(p^2+q^2)^2}{(p^2-q^2)^2}$.

33. If $\frac{x+y}{p+q}=\frac{y}{q}$, then $\frac{x(x+2y)}{y^2}=\frac{p(p+2q)}{q^2}$.

34. If $\frac{x+y+p+q}{x-y+p-q}=\frac{x+y-p-q}{x-y-p+q}$, then $\frac{x+p}{y+q}=\frac{x-p}{y-q}$

35. If $\frac{x+y-p+q}{x-y+p+q}=\frac{y+p+x+q}{y-p-x+q}$, then $\frac{x+q}{y-p}=\frac{y+q}{p+x}$.

36. If $\frac{x^3+3xy^2}{3x^2y+y^3}=\frac{p^3+3pq^2}{3p^2q+q^3}$, then $\frac{x+y}{x-y}=\frac{p+q}{p-q}$.

Example 7. If $a:b=b:c$, then $\frac{a^2+ab+b^2}{b^2+bc+c^2}=\frac{a^2}{b^2}$.

Since $\frac{a}{b}=\frac{b}{c}$ $\quad\therefore\quad b^2=ac.$

$$\therefore\quad \frac{a^2+ab+b^2}{b^2+bc+c^2}=\frac{a^2+ab+ac}{ac+bc+c^2}=\frac{a(a+b+c)}{c(a+b+c)}=\frac{a}{c}$$

$$=\frac{a}{b}\times\frac{b}{c}=\frac{a}{b}\times\frac{a}{b}=\frac{a^2}{b^2}.$$

NOTE. Putting each of the ratios $=k$, we get $b=ck$ and $a=bk=ck^2$. Using these results, we can do this example by the ordinary **k-method.**

If $\frac{a}{b}=\frac{b}{c}=\frac{c}{d}$, prove that

37. $\frac{a^2+b^2+c^2}{b^2+c^2+d^2}=\frac{ab+bc+c^2}{bc+cd+d^2}$.

38. $\frac{a^2+b^2+c^2}{b^2+c^2+d^2}=\frac{a^2+bc}{b^2+cd}$.

39. $\frac{a^3+b^3+c^3}{b^3+c^3+d^3}=\frac{a}{d}$.

40. $\left(\frac{a-b}{b-c}\right)^3=\frac{a}{d}$.

41. $a-2b+c=\frac{(a-b)^2}{a}=\frac{(b-c)^2}{c}$.

42. $\frac{(a+b+c)^2}{a^2+b^2+c^2}=\frac{a+b+c}{a-b+c}$.

43. $(b-c)^2=(a-b)(c-d)$.

44. $(a+b+c)(a-b+c)=a^2+b^2+c^2$.

45. $(a-d)^2=(b-c)^2+(c-a)^2+(d-b)^2$.

46. $(a^2+b^2+c^2)(b^2+c^2+d^2)=(ab+bc+cd)^2$.

47. a^2+b^2, b^2+c^2 and c^2+d^2 are in continued proportion.

48. If a, b, c are in continued proportion, find the simplest value of $\dfrac{abc(a+b+c)^3}{(ab+ac+bc)^3}$.

49. If a, b, c are in continued proportion and if $a(b-c)=2b$, prove that $a-c=2\left(\dfrac{a+b}{a}\right)$.

6. Theorem. If $\dfrac{a}{b}=\dfrac{c}{d}=\dfrac{e}{f}=\ldots$ then each of the ratios $=\dfrac{pa+qc+re+}{pb+qd+rf+}\ldots$, where $p, q, r\ldots$ are any quantities whatever.

Let $\dfrac{a}{b}=\dfrac{c}{d}=\dfrac{e}{f}=\ldots=k.$

Then $a=bk,\ c=dk,\ e=fk\ldots$

$\therefore\quad pa=pbk,\ qc=qdk,\ re=rfk\ldots$

By addition we have

$$pa+qc+re+\ldots=k(pb+qd+rf+\ldots)$$

$$\therefore\quad \frac{pa+qc+re+}{pb+qd+rf+\ldots}=k=\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\ldots$$

Cor. 1. Putting $p=q=r=\ldots=1$, we have

$$\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\ldots=\frac{a+c+e+\ldots}{b+d+f+\ldots}.$$

Thus, *when several fractions are equal, each of them is equal to the sum of all the numeraiors divided by the sum of all the denominators.*

Cor. 2. Putting $p=1$ and $q=-1$, we have

$$\frac{a}{b}=\frac{c}{d}=\ldots=\frac{a-c}{b-d}.$$

Thus, *when two fractions are equal, each is equal to the first numerator minus the second numerator divided by the first denominator minus the second denominator.*

EXERCISE 62.

If $a : b = c : d = e : f$, then

1. $\dfrac{a^2+c^2+e^2}{b^2+d^2+f^2} = \dfrac{a^2}{b^2}$.

2. $\dfrac{a^3-2c^3+3e^3}{b^3-2d^3+3f^3} = \dfrac{a^3}{b^3}$.

3. $\dfrac{3a^2-4c^2+5e^2}{3b^2-4d^2+5f^2} = \dfrac{ac}{bd}$.

4. $\dfrac{3a+4c+2e}{3b+4d+2f} = \dfrac{2a-3c-4e}{2b-3d-4f}$.

5. $\dfrac{a^3+c^3+e^3}{b^3+d^3+f^3} = \dfrac{a^2c+c^2e+e^2a}{b^2d+d^2f+f^2b}$.

6. $(a^2+c^2+e^2)(b^2+d^2+f^2) = (ab+cd+ef)^2$.

7 If $\dfrac{4x+5}{6x+7} = \dfrac{4x+6}{6x+4}$, then each $= -\dfrac{1}{3}$.

8. If $\dfrac{1-4x}{1+6x} = \dfrac{3+2x}{2-3x}$, then each $= \dfrac{7}{5}$.

9. If $\dfrac{x^2+3x+2}{x^2+7x+12} = \dfrac{x+3}{x+7}$, then each $= \dfrac{1}{6}$.

10. If $\left(\dfrac{2x+3}{2x+5}\right)^2 = \dfrac{x+3}{x+5}$, then $\dfrac{9}{x+3} = \dfrac{25}{x+5}$.

Fill up the blanks in the following:

11. $\dfrac{x}{8} = \dfrac{y}{3} = \dfrac{x+y}{\quad} = \dfrac{x-y}{\quad} = \dfrac{15x-12y}{\quad}$.

12. $\dfrac{x}{7} = \dfrac{y}{3} = \dfrac{\quad}{10} = \dfrac{\quad}{4} = \dfrac{\quad}{36}$.

13. If $\dfrac{x}{a} = \dfrac{y}{b} = \dfrac{z}{c}$, shew that $\left(\dfrac{ax+by+cz}{a^2+b^2+c^2}\right)^3 = \dfrac{xyz}{abc}$.

14. If a, b, c, d are in proportion, prove that $ab+cd$ is the mean proportional between a^2+c^2 and b^2+d^2; and if a, b, c, d are in continued proportion, prove that $b+c$ is the mean proportional between $a+b$ and $c+d$.

7. A few important Examples.

Example 1. If $\frac{x}{b+c-a}=\frac{y}{c+a-b}=\frac{z}{a+b-c}$, find the value of $(b-c)x+(c-a)y+(a-b)z$.

Let $\frac{x}{b+c-a}=\frac{y}{c+a-b}=\frac{z}{a+b-c}=k.$

Then $x=(b+c-a)k$
$y=(c+a-b)k$
$z=(a+b-c)k$

$\therefore$ the given expression
$=(b-c)(b+c-a)k+(c-a)(c+a-b)k+(a-b)(a+b-c)k$
$=k\{b^2-c^2-a(b-c)+c^2-a^2-b(c-a)+a^2-b^2-c(a-b)\}$
$=k\times 0=0.$

Example 2. If $\frac{a}{b+c}=\frac{b}{c+a}=\frac{c}{a+b}$, then each $=\frac{1}{2}$ or -1.

$$\text{Each ratio}=\frac{\text{sum of the numerators}}{\text{sum of the denominators}}$$
$$=\frac{a+b+c}{2a+2b+2c}$$
$$=\frac{a+b+c}{2(a+b+c)}$$
$=\frac{1}{2}$, if $a+b+c$ *is not equal to* 0.

When $a+b+c=0$, then $b+c=-a$

$\therefore$ each ratio $=\frac{a}{b+c}=\frac{a}{-a}=-1.$

Example 3. If $\frac{x}{b+c}=\frac{y}{c+a}=\frac{z}{a+b}$,

then $\frac{a}{y+z-x}=\frac{b}{z+x-y}=\frac{c}{x+y-z}.$

Let $\frac{x}{b+c}=\frac{y}{c+a}=\frac{z}{a+b}=k.$

Then $x=(b+c)k$, $y=(c+a)k$ and $z=(a+b)k$.

$\therefore$ $y+z-x=(c+a)k+(a+b)k-(b+c)k$
$=k(c+a+a+b-b-c)$
$=2ka$

$\therefore$ $\frac{a}{y+z-x}=\frac{1}{2k}.$

Similarly, $\frac{b}{z+x-y}=\frac{1}{2k}$ and $\frac{c}{x+y-z}=\frac{1}{2k}$

$$\therefore \quad \frac{a}{y+z-x}=\frac{b}{z+x-y}=\frac{c}{x+y-z}.$$

Example 4. If $\frac{ay-bx}{c}=\frac{cx-az}{b}=\frac{bz-cy}{a}$, shew that $\frac{x}{a}=\frac{y}{b}=\frac{z}{c}$.

Let each of the given ratios $=k$.

Then $ay-bx=ck$, $cx-az=bk$, $bz-cy=ak$

$\therefore \quad c(ay-bx)=c^2k$, $b(cx-az)=b^2k$, $a(bz-cy)=a^2k$.

Adding these results, we get

$$k(a^2+b^2+c^2)=0$$

$\therefore \quad k=0$

$\therefore \quad ay-bx=0$ or $ay=bx$, $\quad \therefore \quad \frac{x}{a}=\frac{y}{b}$,

also $\quad cx-az=0$ or $cx=az$, $\quad \therefore \quad \frac{x}{a}=\frac{z}{c}$.

$\therefore \quad \frac{x}{a}=\frac{y}{b}=\frac{z}{c}$.

EXERCISE 63.

1. If $\frac{x}{a-b}=\frac{y}{b-c}=\frac{z}{c-a}$, prove that $x+y+z=0$ and $cx+ay+bz=0$.

2. If $\frac{x}{bc(b-c)}=\frac{y}{ca(c-a)}=\frac{z}{ab(a-b)}$, prove that $a(b+c)x+b(c+a)y+c(a+b)z=0$.

3. If $\frac{a}{b+c-a}=\frac{b}{c+a-b}=\frac{c}{a+b-c}$, prove that each ratio $=1$.

4. If $\frac{a}{b+c}=\frac{b}{c+a}=\frac{c}{a+b}$ and $a+b+c$ be not equal to zero, then $a=b=c$.

*5. Assuming that $\frac{a+b-c}{a+b}=\frac{b+c-a}{b+c}=\frac{c+a-b}{c+a}$ and $a+b+c$ is not zero, shew that $a=b=c$.

*6. If $\frac{a+b}{b+c}=\frac{c+d}{d+a}$, prove that either $a=c$ or $a+b+c+d=0$.

*7. If $\frac{x}{a+2b+2c}=\frac{y}{b+2c+2a}=\frac{z}{c+2a+2b}$, shew that each of these ratios $=\frac{x+y-z}{a+b+3c}$.

*8. If $\frac{x}{ax+by+cz}=\frac{y}{bx+cy+az}=\frac{z}{cx+ay+bz}$, shew that each of these ratios $=\frac{1}{a+b+c}$, provided $x+y+z$ is not equal to zero.

*9. If $(a+b+c)x=(b+c-a)y=(c+a-b)z=(a+b-c)w$ then $\frac{1}{y}+\frac{1}{z}+\frac{1}{w}=\frac{1}{x}$.

*10. If $a+c=2b$ and $\frac{1}{b}+\frac{1}{d}=\frac{2}{c}$, shew that $a:b=c:d$.

CHAPTER XIV

FRACTIONAL AND LITERAL EQUATIONS

1. Equations involving fractions which contain the unknown in the denominators.

Example 1. Solve $\frac{1}{x}-\frac{3}{2x}+\frac{5}{3x}-\frac{7}{4x}=\frac{5}{6x}-\frac{1}{24x}-1$.

Multiplying both sides by $24x$, the L.C.M. of the denominators, we have

$$24-36+40-42=20-1-24x.$$

By transposition, we get

$$24x=20-1-24+36-40+42$$

or $24x=33$

$$\therefore \quad x=1\frac{3}{8}.$$

Example 2. Solve $\frac{2}{3x-4}-\frac{3}{4x-5}=0$.

Multiplying both sides by $(3x-4)(4x-5)$, the L.C.M. of the denominators, we have

$$2(4x-5)-3(3x-4)=0$$

$$8x-10-9x+12=0$$

or $$-x=-2$$

or $$x=2.$$

Otherwise

By transposition, we have

$$\frac{2}{3x-4}=\frac{3}{4x-5}.$$

Multiplying both sides by $(3x-4)(4x-5)$, the L.C.M. of the denominators, or *multiplying cross-wise*, we have

$$2(4x-5)=3(3x-4)$$

or $$8x-10=9x-12$$

or $$x=2.$$

EXERCISE 64.

Solve the equations:

1. $\frac{1}{2x}+\frac{1}{3x}-2=\frac{4}{3x}-3.$

2. $\frac{1}{x}+\frac{1}{2x}+\frac{1}{3x}+\frac{1}{4x}=6.$

3. $\frac{\cdot 4}{x}=\cdot 25-\frac{\cdot 65}{x}.$

4. $\frac{3}{2x}+\frac{4}{5x}+11=\frac{1}{x}+\frac{5}{3x}.$

5. $\frac{a}{bx}-\frac{b}{ax}=a^2-b^2.$

6. $\frac{1}{5x-11}=\frac{2}{3x-1}.$

7. $\frac{5}{3x+4}=\frac{4}{5(x-3)}.$

8. $\frac{3}{2x+3}=\frac{4}{3x+2}.$

9. $\frac{5}{3-4x}=\frac{6}{6-5x}.$

10. $\frac{2}{3x-1}+\frac{3}{2x-1}=0.$

11. $\frac{3}{8x+5}=\frac{5}{6x-47}=0.$

12. $\frac{a}{a-x}+\frac{b}{b-x}=0.$

13. $\frac{a}{bx+a}+\frac{b}{ax+b}=0.$

14. $\frac{3}{6x^2+2x+5}=\frac{4}{8x^2+3x+2}.$

15. $\frac{a}{ax^2+bx+1}=\frac{b}{bx^2+ax+1}.$

16. $\frac{2}{4x^3+2x^2+x+1}=\frac{3}{6x^3+3x^2+2x+1}.$

Example 3. Solve $\frac{1}{(x-1)(x-2)}+\frac{1}{(x-1)(x-3)}$

$$=\frac{1}{(x-2)(x-3)}+\frac{1}{(x-3)(x-4)}.$$

By transposition, we have

$$\frac{1}{(x-1)(x-2)}-\frac{1}{(x-2)(x-3)}=\frac{1}{(x-3)(x-4)}-\frac{1}{(x-1)(x-3)}$$

$$\therefore \quad \frac{x-3-x+1}{(x-1)(x-2)(x-3)}=\frac{x-1-x+4}{(x-1)(x-3)(x-4)}$$

$$\therefore \quad \frac{-2}{(x-1)(x-2)(x-3)}=\frac{3}{(x-1)(x-3)(x-4)}$$

Multiplying both sides by $(x-1)(x-3)$, we have

$$\frac{-2}{x-2}=\frac{3}{x-4}.$$

Multiplying cross-wise, we have

$$-2x+8=3x-6$$

$$5x=14$$

$$\therefore \quad x=2\tfrac{4}{5}.$$

or

NOTE. In the above process, the first step of transposition is taken to remove x from the numerators on both sides, when they are simplified.

Example 4. Solve $\dfrac{10}{2x+1}+\dfrac{3}{3x-2}=\dfrac{2}{2x-3}+\dfrac{15}{3x+2}$

Here, we arrange the terms in such a way as to remove x from the numerator on both sides, when they are simplified.

By transposition, we have

$$\frac{10}{2x+1}-\frac{15}{3x+2}=\frac{2}{2x-3}-\frac{3}{3x-2}$$

$$\therefore \quad \frac{30x+20-30x-15}{(2x+1)(3x+2)}=\frac{6x-4-6x+9}{(2x-3)(3x-2)}$$

$$\therefore \quad \frac{5}{(2x+1)(3x+2)}=\frac{5}{(2x-3)(3x-2)}$$

$$\therefore \quad \frac{1}{(2x+1)(3x+2)}=\frac{1}{(2x-3)(3x-2)}.$$

Multiplying cross-wise, we have

$$(2x-3)(3x-2)=(2x+1)(3x+2)$$

$$\therefore \quad 6x^2-13x+6=6x^2+7x+2$$

$$\therefore \quad 20x=4$$

$$\therefore \quad x=\tfrac{1}{5}.$$

Solve the equations:

17. $\dfrac{1}{x+1}-\dfrac{1}{x+3}=\dfrac{1}{x+2}-\dfrac{1}{x+4}.$

18. $\dfrac{1}{x+2}+\dfrac{1}{x+10}=\dfrac{1}{x+4}+\dfrac{1}{x+8}.$

19. $\dfrac{1}{x}-\dfrac{1}{x-1}+\dfrac{1}{x+3}-\dfrac{1}{x+4}=0.$

20. $\dfrac{1}{x-6}-\dfrac{1}{x-3}=\dfrac{1}{x-5}-\dfrac{1}{x-2}.$

21. $\dfrac{1}{x-10}-\dfrac{1}{x-5}=\dfrac{1}{x-7}-\dfrac{1}{x-2}.$

22. $\dfrac{1}{x^2+7x+12}+\dfrac{1}{x^2+8x+15}=\dfrac{1}{x^2+9x+18}+\dfrac{1}{x^2+9x+20}$.

23. $\dfrac{1}{(x+a)^2-b^2}+\dfrac{1}{(x+b)^2-a^2}=\dfrac{1}{x^2-(a+b)^2}+\dfrac{1}{x^2-(a-b)^2}$.

24. $\dfrac{2}{2x-1}+\dfrac{3}{3x+1}=\dfrac{2}{2x+1}+\dfrac{3}{3x-2}$.

25. $\dfrac{9}{x-4}+\dfrac{3}{x-8}=\dfrac{4}{x-9}+\dfrac{8}{x-3}$.

26. $\dfrac{3}{x-3}-\dfrac{4}{x+9}-\dfrac{5}{x-27}+\dfrac{6}{x-15}=0$.

27. $\dfrac{6}{5x+7}-\dfrac{4}{5x+13}=\dfrac{9}{5x+13}-\dfrac{7}{5x+19}$.

28. $\dfrac{1}{2x-1}+\dfrac{8}{4x-1}-\dfrac{6}{3x-1}-\dfrac{3}{6x-1}=0$.

Example 5. Solve $\dfrac{1}{x+5}+\dfrac{10}{2x-5}=\dfrac{18}{3x-5}$.

Method. Decompose $\dfrac{18}{3x-5}$ into two such fractions that when one of them is combined with $\dfrac{1}{x+5}$ and the other with $\dfrac{10}{2x-5}$ and then simplified, x disappears from the numerators on both sides.

Thus, $\dfrac{1}{x+5}+\dfrac{10}{2x-5}=\dfrac{3+15}{3x-5}=\dfrac{3}{3x-5}+\dfrac{15}{3x-5}$.

Combining $\dfrac{3}{3x-5}$ with $\dfrac{1}{x+5}$ and $\dfrac{15}{3x-5}$ with $\dfrac{10}{2x-5}$,

we have $\dfrac{1}{x+5}-\dfrac{3}{3x-5}=\dfrac{15}{3x-5}-\dfrac{10}{2x-5}$

$\therefore \dfrac{3x-5-3x-15}{(x+5)(3x-5)}=\dfrac{30x-75-30x+50}{(3x-5)(2x-5)}$

$\therefore \dfrac{-20}{(x+5)(3x-5)}=\dfrac{-25}{(3x-5)(2x-5)}$.

Multiplying both sides by $\frac{3x-5}{-5}$ we have

$$\frac{4}{x+5}=\frac{5}{2x-5}.$$

Multiplying cross-wise we have

$$8x-20=5x+25$$

$$\therefore \quad 3x=45$$

$$\therefore \quad x=15.$$

Example 6. Solve $\frac{6}{2x+3}-\frac{5}{5x+2}=\frac{2}{x+6}$.

Method. Decompose $\frac{2}{x+6}$ into two such fractions that when one is combined with $\frac{6}{2x+3}$ and the other with $\frac{5}{5x+2}$ and then simplified, x disappears from the numerators on both sides. Thus,

$$\frac{6}{2x+3}-\frac{5}{5x+2}=\frac{3-1}{x+6}=\frac{3}{x+6}-\frac{1}{x+6}.$$

Combining $\frac{3}{x+6}$ with $\frac{6}{2x+3}$ and $\frac{1}{x+6}$ with $\frac{5}{5x+2}$, we have

$$\frac{6}{2x+3}-\frac{3}{x+6}=\frac{5}{5x+2}-\frac{1}{x+6}$$

$$\therefore \quad \frac{6x+36-6x-9}{(2x+3)(x+6)}=\frac{5x+30-5x-2}{(5x+2)(x+6)}$$

$$\therefore \quad \frac{27}{(2x+3)(x+6)}=\frac{28}{(5x+2)(x+6)}.$$

Multiplying both sides by $x+6$, we have

$$\frac{27}{2x+3}=\frac{28}{5x+2}.$$

Multiplying cross-wise, we have

$$27(5x+2)=28(2x+3)$$

or $$135x+54=56x+84$$

$$\therefore \quad 79x=30$$

$$\therefore \quad x=\tfrac{30}{79}.$$

Solve the equations :

29. $\frac{1}{x+1}+\frac{2}{x+2}=\frac{3}{x+3}$.

30. $\frac{3}{x+1}+\frac{4}{x+2}=\frac{7}{x+3}$.

31. $\frac{2}{2x-5}+\frac{1}{x-3}=\frac{6}{3x-1}$.

32. $\frac{7}{x+2}+\frac{3}{x+4}=\frac{10}{x-5}$.

33. $\frac{1}{x+a}+\frac{1}{x+b}=\frac{2}{x}$.

34. $\frac{a}{x-1}+\frac{b}{x-2}=\frac{a+b}{x}$.

35. $\frac{a}{x-a}+\frac{b}{x-b}=\frac{a+b}{x-c}$.

36. $\frac{a}{x+a}-\frac{b}{x+b}=\frac{a-b}{x+c}$.

37. $\frac{3}{4x+1}+\frac{4}{4x+5}=\frac{7}{4x+3}$.

38. $\frac{6}{3x+15}-\frac{1}{x-5}=\frac{2}{2x-5}$.

39. $\frac{4}{5x+1}+\frac{7}{10x+1}=\frac{3}{2x+1}$.

40. $\frac{6}{5-6x}+\frac{13}{6x+19}=\frac{7}{6x+7}$.

41. $\frac{6}{2x-3}+\frac{2}{1-2x}=\frac{2}{x-1}$.

42. $\frac{10}{5x-9}+\frac{14}{2x+9}=\frac{9}{x+8}$.

43. $\frac{7}{7x+1}+\frac{6}{4x+1}=\frac{15}{6x+5}$.

44. $\frac{b-c}{x+a}+\frac{a-b}{x+b}=\frac{a-c}{x+c}$.

45. $\frac{1}{x+2}+\frac{3}{x+3}+\frac{5}{x+5}=\frac{9}{x+4}$.

[*Hint.* Decompose $\frac{9}{x+4}$ into three parts with numerators 1, 3, 5, and then associate these parts with the fractions on the left-hand side so that x may disappear from the numerators after simplification.]

46. $\frac{1}{x-6a}+\frac{2}{x+3a}+\frac{3}{x-2a}=\frac{6}{x-a}$.

47. $\frac{3}{3x+1}+\frac{4}{4x+1}+\frac{48}{8x+1}=\frac{48}{6x+1}$.

2. Equations involving fractions which contain the unknown both in the numerators and the denominators.

Example 1. Solve $\frac{4x+1}{5x+2}=\frac{2}{3}$.

Multiplying cross-wise, we have

$$3(4x+1)=2(5x+2)$$

$\therefore$ $$12x+3=10x+4$$

$\therefore$ $$2x=1 \text{ and } x=\tfrac{1}{2}.$$

Example 2. Solve $\frac{4x+5}{6x+7}=\frac{2x+3}{3x+2}$.

Multiplying cross-wise, we have

$$(4x+5)(3x+2)=(2x+3)(6x+7)$$

$$\therefore \quad 12x^2+23x+10=12x^2+32x+21$$

$$\therefore \quad -9x=11$$

$$\therefore \quad x=-1\tfrac{2}{9}.$$

EXERCISE 65.

Solve the equations :

1. $\frac{5x+2}{3x-4}=3\frac{2}{5}$.
2. $\frac{7x-3}{2x+5}=1\frac{2}{9}$.
3. $\frac{4x+5}{6x-7}=1\frac{6}{11}$.
4. $\frac{2-3x}{1-4x}+\frac{1}{2}=0$.
5. $\frac{11-5x}{7-4x}=2$.
6. $\frac{ax+b}{bx+a}=\frac{c}{d}$.
7. $\frac{2x+1}{3x+2}=\frac{2x+5}{3x-4}$.
8. $\frac{3x+2}{3x+1}=\frac{4x-3}{4x+1}$.
9. $\frac{3-5x}{7-3x}=\frac{5x+2}{3x+4}$.
10. $\frac{7-12x}{5-2x}=\frac{6x-5}{x-3}$.
11. $\frac{x^2+x+1}{x^2+3x+3}=\frac{x+1}{x+3}$.
12. $\frac{ax+b^2}{ax+c^2}=\frac{x+a}{x+b}$.

Example 3. Solve $\frac{x^2+2x-3}{x^2-9}=3$.

$$\text{The left-hand side}=\frac{(x+3)(x-1)}{(x+3)(x-3)}=\frac{x-1}{x-3}$$

$$\therefore \quad \frac{x-1}{x-3}=3.$$

Multiplying cross-wise, we have

$$3x-9=x-1$$

$$\therefore \quad x=4.$$

Solve the equations :

13. $\frac{x^2+x-6}{x^2-4}=1\frac{1}{5}$.
14. $\frac{x^2+x-12}{x^2-16}=2$.

15. $\dfrac{x^2-7x+12}{x^2-9}=\frac{2}{9}$.

16. $\dfrac{x^2-x-20}{x^2+x-12}=\frac{1}{2}$.

17. $\dfrac{x^2+4x-21}{x^2+3x-28}=1\frac{1}{4}$.

18. $\dfrac{x^2+x-30}{x^2+3x-18}=\frac{1}{2}$.

Example 4. Solve $\dfrac{4x+3}{9}+\dfrac{29-7x}{12-5x}=\dfrac{8x+19}{18}$.

Transposing the terms with numerical denominators to one side, we have

$$\frac{29-7x}{12-5x}=\frac{8x+19}{18}-\frac{4x+3}{9}$$

$$=\frac{8x+19-8x-6}{18}$$

$$=\frac{13}{18}.$$

Multiplying cross-wise, we have

$$18(29-7x)=13(12-5x)$$

$\therefore$ $$522-126x=156-65x$$

$$61x=366 \text{ and } x=6.$$

Solve the equations :

19. $\dfrac{x-3}{4}+\dfrac{x+1}{12x+7}=\dfrac{3x-11}{12}$.

20. $\dfrac{2x-1}{5}+\dfrac{3x+1}{15x+8}=\dfrac{8x-7}{20}$.

21. $\dfrac{x+4}{3}+\dfrac{3x+2}{2x+5}=\dfrac{3x+16}{9}$.

22. $\dfrac{x-a}{b}+\dfrac{2x+a}{x+b}=\dfrac{ax+b^2}{ab}$.

23. $\dfrac{x}{2}+\dfrac{8x-12}{10x-4}=\dfrac{18x+20}{36}$.

24. $\dfrac{91x-21}{56}+\dfrac{24x-93}{35x-138}=\dfrac{13x+9}{8}$.

Example 5. Solve $\dfrac{4x-13}{2x-5}+\dfrac{9x-17}{3x-7}=5$.

Decomposing the terms on the left-hand side into integral and fractional parts, we have

$$\frac{4x-10-3}{2x-5}+\frac{9x-21+4}{3x-7}=5$$

or $$\frac{2(2x-5)-3}{2x-5}+\frac{3(3x-7)+4}{3x-7}=5$$

or $$2-\frac{3}{2x-5}+3+\frac{4}{3x-7}=5$$

$$\therefore \quad -\frac{3}{2x-5}+\frac{4}{3x-7}=0$$

By transposition, we have

$$\frac{4}{3x-7}=\frac{3}{2x-5}.$$

Multiplying cross-wise, we have

$$8x-20=9x-21$$

$$\therefore \quad x=1.$$

Example 6. Solve $\frac{6x-1}{3x+4}+\frac{25x-40}{5x-6}=\frac{7x+9}{x+2}$.

Decomposing all the terms into integral and fractional parts, we have

$$\frac{2(3x+4)-9}{3x+4}+\frac{5(5x-6)-10}{5x-6}-\frac{7(x+2)-5}{x+2}$$

or $$2-\frac{9}{3x+4}+5-\frac{10}{5x-6}=7-\frac{5}{x+2}$$

$$\frac{9}{3x+4}+\frac{10}{5x-6}=\frac{5}{x+2}$$

$$=\frac{3+2}{x+2}$$

$$=\frac{3}{x+2}+\frac{2}{x+2}.$$

By transposition, we have

$$\frac{9}{3x+4}-\frac{3}{x+2}=\frac{2}{x+2}-\frac{10}{5x-6}$$

$$\therefore \quad \frac{9x+18-9x-12}{(3x+4)(x+2)}=\frac{10x-12-10x-20}{(x+2)(5x-6)}$$

$$\therefore \quad \frac{6}{(3x+4)(x+2)}=\frac{-32}{(x+2)(5x-6)}.$$

Multiplying both sides by $\frac{x+2}{2}$, we have

$$\frac{3}{3x+4}=\frac{-16}{5x-6}.$$

Multiplying cross-wise, we have

$$15x-18=-48x-64$$

$$\therefore \quad 63x=-46$$

$$\therefore \quad x=-\frac{46}{63}.$$

Solve the equations:

25. $\frac{3x+1}{x-1}+\frac{2x+3}{x+1}=5.$

26. $\frac{8x-7}{2x+1}+\frac{6x+1}{3x-2}=6.$

27. $\frac{5-2x}{4-x}-\frac{3-x}{1-x}=1.$

28. $\frac{5-4x}{3-2x}+\frac{2-3x}{1-x}=5.$

29. $\frac{x^2+5x+6}{x+1}+\frac{x^2+3x+5}{x+2}=2x+5.$

[*Hint.* Decompose these fractions by division.]

30. $\frac{2x^2-9x-8}{2x+1}+\frac{x^2-7x+15}{x-3}=2x-9.$

31. $\frac{4x+3}{4x-3}+\frac{15x-7}{5x-4}=\frac{8x+1}{2x-1}.$

32. $\frac{6x+1}{2x-1}+\frac{6x+7}{3x-1}=\frac{25x+20}{5x-1}.$

33. $\frac{x+2}{x+1}+\frac{x+3}{x+2}=\frac{2x+8}{x+3}.$

34. $\frac{15x+11}{3x+4}+\frac{12x+5}{4x+3}=\frac{8x+44}{x+6}.$

35. $\frac{6x-5}{2x-3}-\frac{6x-14}{3x-1}=\frac{4x+23}{4x-1}.$

36. $\frac{42x-37}{6x-1}+\frac{20x+13}{5x+12}=\frac{11x+76}{x+8}.$

37. $\frac{2x+7}{x+2}+\frac{3x+13}{x+3}=\frac{5x+27}{x+4.}.$

38. $\frac{4x-7}{4x+5}+\frac{15x+11}{5x+7}=\frac{12x+1}{3x+4}.$

39. $\frac{x^2+4}{x-1}+\frac{x^2-x+2}{x-2}=\frac{2x^2-4x+3}{x-3}.$

40. $\frac{2x^2+x+5}{2x-1}+\frac{3x^2-2x+20}{3x+1}=\frac{2x^2+6x+10}{x+3}.$

41. $\frac{x}{2x-a}+\frac{x}{2x-b}=1.$

[*Hint.* Multiply both sides by 2 and proceed as before.

42. $\frac{3x+5}{x+1}=\frac{4x+8}{3x+3}+\frac{10x+1}{6x+3}.$

***Example 7.** Solve $\frac{2x+7}{2x+3}+\frac{6x+11}{6x-1}=\frac{3x+7}{3x+1}+\frac{x+3}{x+1}.$

Decomposing each term into integral and fractional parts, we have

$$\frac{(2x+3)+4}{2x+3}+\frac{(6x-1)+12}{6x-1}=\frac{(3x+1)+6}{3x+1}+\frac{(x+1)+2}{x+1}$$

$$\therefore\left(1+\frac{4}{2x+3}\right)+\left(1+\frac{12}{6x-1}\right)=\left(1+\frac{6}{3x+1}\right)+\left(1+\frac{2}{x+1}\right),$$

$$\therefore\quad \frac{4}{2x+3}+\frac{12}{6x-1}=\frac{6}{3x+1}+\frac{2}{x+1}.$$

Taking two terms on one side and two on the other, in such a way as to remove x from the numerators on both sides when they are simplified, we have

$$\frac{4}{2x+3}-\frac{6}{3x+1}=\frac{2}{x+1}-\frac{12}{6x-1}$$

$$\therefore\quad \frac{12x+4-12x-18}{(2x+3)(3x+1)}=\frac{12x-2-12x-12}{(x+1)(6x-1)}$$

$$\therefore\quad \frac{-14}{(2x+3)(3x+1)}=\frac{-14}{(x+1)(6x-1)}.$$

Dividing both sides by -14 and then multiplying crosswise, we have

$$(x+1)(6x-1)=(2x+3)(3x+1)$$
$$\therefore\quad 6x^2+5x-1=6x^2+11x+3$$
$$\therefore\quad 6x=-4$$
$$\therefore\quad x=-\tfrac{2}{3}.$$

Solve the equations:

*43. $\frac{x+1}{x+2}-\frac{x+2}{x+3}=\frac{x+4}{x+5}-\frac{x+5}{x+6}.$

*44. $\frac{2x-11}{x-2}+\frac{x+4}{x-3}=\frac{x-5}{x+2}+\frac{2x+9}{x+1}.$

*45. $\frac{7x-55}{x-8}+\frac{2x-17}{x-9}=\frac{6x-71}{x-12}+\frac{3x-14}{x-5}$.

*46. $\frac{x-8}{x-10}-\frac{x-5}{x-7}=\frac{x-7}{x-9}-\frac{x-4}{x-6}$.

*47. $\frac{2x-3}{x-4}-\frac{2x+1}{x-2}=5\left\{\frac{x-2}{x-1}-\frac{x-4}{x-3}\right\}$.

*48. $\frac{4(x+1)}{2x-3}+\frac{2x+1}{x+3}=\frac{4(x+2)}{2x-1}+\frac{2x+3}{x+4}$.

*49. $\frac{x}{x-2}+\frac{9-x}{7-x}=\frac{x+1}{x-1}+\frac{8-x}{6-x}$.

*50. $\frac{2x-3}{x-2}+\frac{3x-20}{x-7}=\frac{x-3}{x-4}+\frac{4x-19}{x-5}$.

*51. $\frac{2x+11}{x+5}-\frac{9x-9}{3x-4}=\frac{4x+13}{x+3}-\frac{15x-47}{3x-10}$.

*52. $\frac{x^2+4x+5}{x+2}+\frac{x^2+10x+26}{x+5}=\frac{x^2+10x+25}{x+6}+\frac{x^2+4x+4}{x+1}$.

*53. $\frac{x^2-4x-20}{x-7}+\frac{x^2+7x+11}{x+5}=\frac{x^2+3x+3}{x+2}+\frac{x^2-15}{x-4}$.

*3. Solution of fractional equations by the application of the fundamental principles of proportion.

***Example 1.** Solve $\frac{4x+5}{6x+7}=\frac{2x+3}{3x+2}$.

The equations of this type can be easily solved by multiplying cross-wise, but the method illustrated below is considered to be neater.

Multiplying the numerator and the denominator of the term on the right-hand side by 2 and rewriting the equation,

$$\frac{4x+5}{6x+7}=\frac{4x+6}{6x+4}$$

$$\therefore \text{ each} = \frac{(4x+5)-(4x+6)}{(6x+7)-(6x+4)} = -\tfrac{1}{3}$$

$$\therefore \quad \frac{4x+6}{6x+4} \text{ or } \frac{2x+3}{3x+2} = -\tfrac{1}{3}$$

or $\quad 6x+9=-3x-2$

or $\quad 9x=-11$

$\therefore \quad x=-1\tfrac{2}{9}$.

***Example 2.** Solve $\frac{1-4x}{1+6x}=\frac{3+2x}{2-3x}$.

Multiplying the numerator and the denominator of the term on the right-hand side by 2, and rewriting the equation,

$$\frac{1-4x}{1+6x}=\frac{6+4x}{4-6x}$$

$$\therefore \quad \text{each}=\frac{(1-4x)+(6+4x)}{(1+6x)+(4-6x)}=\frac{7}{5}$$

$$\therefore \quad \frac{1-4x}{1+6x}=\frac{7}{5}$$

$$\text{or} \quad 7+42x=5-20x$$

$$\therefore \quad 62x=-2$$

$$\therefore \quad x=-\frac{1}{31}.$$

EXERCISE 66.

Solve the equations :

*1. $\frac{6x+1}{4x+3}=\frac{3x-5}{2x-7}$.

*2. $\frac{9x-5}{6x-1}=\frac{3x-2}{2x-3}$.

*3. $\frac{8x-7}{12x+5}=\frac{2x-3}{3x+4}$.

*4. $\frac{3-4x}{1+6x}=\frac{1+2x}{2-3x}$.

*5. $\frac{2+3x}{3-5x}=\frac{1-9x}{4+15x}$.

*6. $\frac{1+2x}{2-3x}=\frac{5-6x}{1+9x}$.

*7. $\frac{x+p}{x+q}=\frac{px+q^2}{px+r^2}$

*8. $\frac{3+bx}{2+abx}=\frac{1-x}{1-ax}$.

***Example 3.** Solve $\frac{(x+1)(x+2)}{(x+3)(x+4)}=\frac{x+3}{x+7}$

Removing the brackets, we have

$$\frac{x^2+3x+2}{x^2+7x+12}=\frac{x+3}{x+7}.$$

Multiplying the numerator and the denominator of the term on the right-hand side by x, we have

$$\frac{x^2+3x+2}{x^2+7x+12}=\frac{x^2+3x}{x^2+7x}$$

$$\therefore \quad \text{each}=\frac{(x^2+3x+2)-(x^2+3x)}{(x^2+7x+12)-(x^2+7x)}=\frac{2}{12}=\frac{1}{6}.$$

$$\therefore \quad \frac{x+3}{x+7}=\frac{1}{6}$$

$$\therefore \quad 6x+18=x+7$$

$$\therefore \quad 5x=-11$$

$$\therefore \quad x=-2\tfrac{1}{5}.$$

Otherwise

By *alternando*, we have

$$\frac{x^2+3x+2}{x+3}=\frac{x^2+7x+12.}{x+7}.$$

Decomposing each fraction into integral and fractional parts, we have

$$x+\frac{2}{x+3}=x+\frac{12}{x+7}$$

$$\therefore \quad \frac{2}{x+3}=\frac{12}{x+7}$$

$$\frac{1}{x+3}=\frac{6}{x+7}$$

$$\therefore \quad 6x+18=x+7$$

$$\therefore \quad x=-2\tfrac{1}{5}.$$

Solve the following equations by both the methods illustrated above :

*9. $\frac{(x+2)(x+3)}{(x+4)(x+5)}=\frac{x+5}{x+9}$.

*10. $\frac{(x+3)(x+4)}{(x+5)(x+6)}=\frac{x+7}{x+11}$.

*11. $\frac{x^2+6x+8}{x^2+8x+15}=\frac{x+6}{x+8}$.

*12. $\frac{x^2+7x+10}{x^2+10x+24}=\frac{x+7}{x+10}$.

*13. $\frac{(x+3)(x+5)}{(x+4)(x+1)}=\frac{x+8}{x+5}$.

*14. $\frac{2-5x-6x^2}{2-2x-3x^2}=\frac{6x+5}{3x+2}$.

***Example 4.** Solve $\left(\frac{2x+3}{2x+5}\right)^2=\frac{x+3}{x+5}$.

By *alternando*, we have

$$\frac{(2x+3)^2}{x+3}=\frac{(2x+5)^2}{x+5}$$

or $$\frac{4x^2+12x+9}{x+3}=\frac{4x^2+20x+25}{x+5}$$

Decomposing each fraction into integral and fractional parts, we have

$$4x+\frac{9}{x+3}=4x+\frac{25}{x+5}$$

$$\therefore \quad \frac{9}{x+3}=\frac{25}{x+5}.$$

Multiplying cross-wise, we have

$$25x+75=9x+45$$

$$\therefore \quad 16x=-30$$

$$\therefore \quad x=-1\tfrac{7}{8}.$$

Solve the equations:

*15. $\left(\frac{x+2}{x+3}\right)^2=\frac{x+4}{x+6}$.

*16. $\left(\frac{x+3}{x+5}\right)^2=\frac{x+6}{x+10}$.

*17. $\left(\frac{x-4}{x-5}\right)^2=\frac{x-8}{x-10}$.

*18. $\left(\frac{2x-3}{2x-5}\right)^2=\frac{x-3}{x-5}$.

*19. $\left(\frac{3x-5}{3x-2}\right)^2=\frac{3x-10}{3x-4}$.

*20. $\left(\frac{x+3}{x+5}\right)^3=\frac{x+1}{x+7}$.

*21. $\left(\frac{ax-b}{ax-c}\right)^2=\frac{ax-2b}{ax-2c}$.

*22. $\left(\frac{ax-b}{ax+c}\right)^3=\frac{ax-2b-c}{ax+2c+b}$.

***Example 5.** Solve $(x^2+3x-7)(x^2+2x-5)$
$=(x^2-3x+7)(x^2-2x+5)$.

Dividing both sides of the equation by $(x^2-3x+7)(x^2+2x-5)$, we have

$$\frac{x^2+3x-7}{x^2-3x+7}=\frac{x^2-2x+5}{x^2+2x-5}.$$

By componendo and dividendo, we have

$$\frac{(x^2+3x-7)+(x^2-3x+7)}{(x^2+3x-7)-(x^2-3x+7)}=\frac{(x^2-2x+5)+(x^2+2x-5)}{(x^2-2x+5)-(x^2+2x-5)}.$$

or $$\frac{2x^2}{6x-14}=\frac{2x^2}{-4x+10}.$$

Dividing both sides by $2x^2$ and multiplying cross-wise, we have

$$-4x+10=6x-14$$

or $$-10x=-24$$

$$\therefore \quad x=\tfrac{12}{5}=2\tfrac{2}{5}.$$

***Example 6.** Solve $16\left(\frac{1-x}{1+x}\right)^3=\frac{1+x}{1-x}$.

Multiplying both sides by $\frac{1-x}{1+x}$, we get

$$16\left(\frac{1-x}{1+x}\right)^4=1.$$

Dividing both sides by 16, we get

$$\left(\frac{1-x}{1+x}\right)^4=\frac{1}{16}=\left(\frac{1}{2}\right)^4$$

$\therefore$ $$\frac{1-x}{1+x}=\tfrac{1}{2}.$$

From this, we get $$x=\tfrac{1}{3}.$$

Solve the equations:

*23. $\frac{x^2-2x+3}{x^2+2x-3}=\frac{x+5}{x-5}$.

*24. $\frac{x^2-3x+4}{3x-4}=\frac{x^2-2x+5}{2x-5}$.

*25. $\frac{x^2+2x-5}{x^2-2x+5}=\frac{x^2-x+3}{x^2+x-3}$.

*26. $(x^2+5x+9)(x^2+3x-7)=(x^2-3x+7)(x^2-5x-9)$.

*27. $(x^2-7x+15)(3x-5)=(x^2-3x+5)(7x-15)$.

*28. $\frac{x^2-2x+5}{3x^2+4x-1}=\frac{x^2+2x-5}{3x^2-4x+1}$.

*29. $81\left(\frac{2-x}{2+x}\right)^3=\left(\frac{2+x}{2-x}\right)$.

*30. $\frac{x^3+3x}{3x^2+1}=\frac{76}{49}$.

4. Literal Equations. The methods *generally* employed in the solution of literal equations are similar to those used in ordinary equations. However, a few typical examples are added below.

Example 1. Solve $a(x-a)+b(x-b)=2ab$.

Removing the brackets, we get

$$ax-a^2+bx-b^2=2ab.$$

By transposition, $ax+bx=a^2+b^2+2ab$

or $$x(a+b)=(a+b)^2$$

$\therefore$ $$x=\frac{(a+b)^2}{a+b}$$

$$=a+b.$$

Example 2. Solve $(x+a)^3+(x+b)^3+(x+c)^3$
$$=3(x+a)(x+b)(x+c).$$

By transposition,

$(x+a)^3+(x+b)^3+(x+c)^3-3(x+a)(x+b)(x+c)=0$

Factorising the left-hand side, we get

$\frac{1}{2}\{(x+a)+(x+b)+(x+c)\}[\{(x+a)-(x+b)\}^2+\{(x+a)$
$-(x+c)\}^2+\{(x+b)-(x+c)\}^2]=0.$

On simplification, we get

$$\tfrac{1}{2}(3x+a+b+c)\{(a-b)^2+(a-c)^2+(b-c)^2\}=0.$$

Either the first factor $=0$ or the second $=0$,

but the second being a *constant* cannot be equal to zero

Hence $(3x+a+b+c)=0$

$\therefore$ $$x=-\frac{a+b+c}{3}.$$

Example 3. Solve $\frac{x+a}{b+c}+\frac{x+b}{c+a}+\frac{x+c}{a+b}+3=0.$

Decomposing 3 into $1+1+1$ and associating *one* with each fraction, we have

$$\left(\frac{x+a}{b+c}+1\right)+\left(\frac{x+b}{c+a}+1\right)+\left(\frac{x+c}{a+b}+1\right)=0$$

or $$\frac{x+a+b+c}{b+c}+\frac{x+a+b+c}{c+a}+\frac{x+a+b+c}{a+b}=0$$

or $$(x+a+b+c)\left\{\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b}\right\}=0.$$

Since the second factor is constant,

$\therefore$ $$x+(a+b+c)=0$$

$\therefore$ $$x=-(a+b+c).$$

***Example 4.** Solve $(x-2a)^3+(x-2b)^3=2(x-a-b)^3$.

Putting p for $(x-2a)$, q for $(x-2b)$ and r for $(x-a-b)$, we get $p^3+q^3=2r^3$

or $p^3-r^3=r^3-q^3$

or $(p-r)(p^2+pr+r^2)=(r-q)(r^2+rq+q^2)$.

But $p-r=r-q$, for each $=b-a$ ... ... (i)

$\therefore$ $p^2+pr+r^2=r^2+rq+q^2$

or $p^2+pr=rq+q^2$

$\therefore$ $p^2-q^2=rq-pr$

or $(p+q)(p-q)=-r(p-q)$

$\therefore$ $p+q=-r.$ ... ... (ii)

Substituting the values of p, q and r in (ii), we get

$(x-2a)+(x-2b)=-(x-a-b)$

whence $3x=3a+3b$

or $x=a+b.$

***Example 5.** Solve $\dfrac{mx+b}{x+a}+\dfrac{nx+a}{x+b}=m+n.$

By transposition, $\left(\dfrac{mx+b}{x+b}-m\right)=\left(n-\dfrac{nx+a}{x+a}\right)$

or $\dfrac{mx+b-mx-mb}{x+b}=\dfrac{nx+na-nx-a}{x+a}$

or $\dfrac{b-mb}{x+b}=\dfrac{na-a}{x+a}.$

Multiplying cross-wise, we have

$(x+a)(b-mb)=(x+b)(na-a)$

or $xb-xmb+ab-abm$

$=xna-xa+abn-ab.$

By transposition,

$xb-xmb-xna+xa=abn-ab-ab+abm$

or $x(b-mb-na+a)=ab(n-2+m)$

$\therefore$ $x=\dfrac{ab(m+n-2)}{a+b-mb-na}.$

EXERCISE 67.

Solve the equations :

1. $a^2(x-a)+b^2(x-b)=abx.$

2. $b(x-2a)+a(x-2b)=(a-b)^2.$

3. $x(x-a)+x(x-b)=2(x-a)(x-b).$

4. $(x-a)^2-(x-b)^2=0.$

5. $(x-a)^2+(x-b)^2=2(x-a-b)^2.$

6. $\frac{a-x}{a}+\frac{2a-x}{2a}=\frac{3a-x}{3a}.$

7. $\frac{a}{bx}+\frac{b}{ax}=a^2+b^2.$

8. $\frac{c-x^2}{bx}-\frac{b-x}{c}=\frac{c-x}{b}-\frac{b-x^2}{cx}.$

9. $\frac{bx}{a}+\frac{ax}{b}+\frac{c^2x}{ab}=a^2+b^2+c^2.$

10. $(x-a)^3+(x-b)^3+(x-c)^3=3(x-a)(x-b)(x-c).$

11. $(x-a)^3+(b-x)^3+(a-b)^3=0.$

12. $\frac{x-b}{a+1}+\frac{x-a}{b+1}+\frac{x-1}{a+b}=3.$

*13. $\frac{x-ab}{b+a}+\frac{x-bc}{b+c}+\frac{x-ca}{c+a}=a+b+c.$

*14. $\frac{bc(ax-1)}{b+c}+\frac{ca(bx-1)}{c+a}+\frac{ab(cx-1)}{a+b}=a+b+c.$

*15. $\frac{x-a^2}{b+c}+\frac{x-ab}{c+a}+\frac{x-ac}{a+b}=3a.$

*16. $\frac{x-a^3}{b^2-bc+c^2}+\frac{x-b^3}{c^2-ca+a^2}+\frac{x-c^3}{a^2-ab+b^2}=2(a+b+c).$

[*Hint.* $2(a+b+c)=(a+b)+(b+c)+(c+a).$]

*17. $\dfrac{a+b}{x-a-b}=\dfrac{a}{x-a}+\dfrac{b}{x-b}$.

*18. $\dfrac{1}{x+a}+\dfrac{1}{x+b}=\dfrac{1}{x+a+b}+\dfrac{1}{x}$.

*19. $\dfrac{ax}{ax-1}+\dfrac{bx}{bx-1}=2$.

*20. $\dfrac{1}{ab-ax}+\dfrac{1}{bc-bx}=\dfrac{1}{ac-ax}$.

*21. $\dfrac{a}{ax+1}+\dfrac{b}{bx+1}=\dfrac{2c}{cx+1}$.

*22. $\dfrac{1}{x+a}+\dfrac{1}{x-a}=\dfrac{2}{x^2-a^2}$.

*23. $\dfrac{ax}{x-b}+\dfrac{bx}{x-a}=a+b$.

*24. $\dfrac{x+2a}{x-2a}-\dfrac{x-2b}{x+2b}=\dfrac{4(a+b)}{x}$.

*25. $\dfrac{ax^2+bx+c}{px^2+qx+r}=\dfrac{ax+b}{px+q}$.

*26. $(x+a+b-c)^3+(x+b+c-a)^3=2(x+b)^3$

*27. $(x-27)^3+(x-23)^3=2(x-25)^3$

*28. $\left(\dfrac{x+a}{x+b}\right)^3=\dfrac{x+2a-b}{x+2b-a}$.

*29. $16\left(\dfrac{a-x}{a+x}\right)^3=\dfrac{a+x}{a-x}$.

CHAPTER XV

SIMULTANEOUS EQUATIONS (*Continued.*)

1. Cross-multiplication Method.

Theorem. If $ax+by+cz=0$ ... ... (i)

and $a'x+b'y+c'z=0$ ... ... (ii)

then $\dfrac{x}{bc'-b'c}=\dfrac{y}{ca'-c'a}=\dfrac{z}{ab'-a'b}$.

Proof. Multiplying (i) by c' and (ii) by c we get

$ac'x+bc'y+cc'z=0$ ... ... (iii)

$a'cx+b'cy+cc'z=0$ ... ... (iv)

Subtracting (iv) from (iii), we have

$(ac'-a'c)x+(bc'-b'c)y=0$

$\therefore\quad (bc'-b'c)y=-(ac'-a'c)x=(ca'-c'a)x.$

Dividing both sides by $(bc'-b'c)(ca'-c'a)$, we have

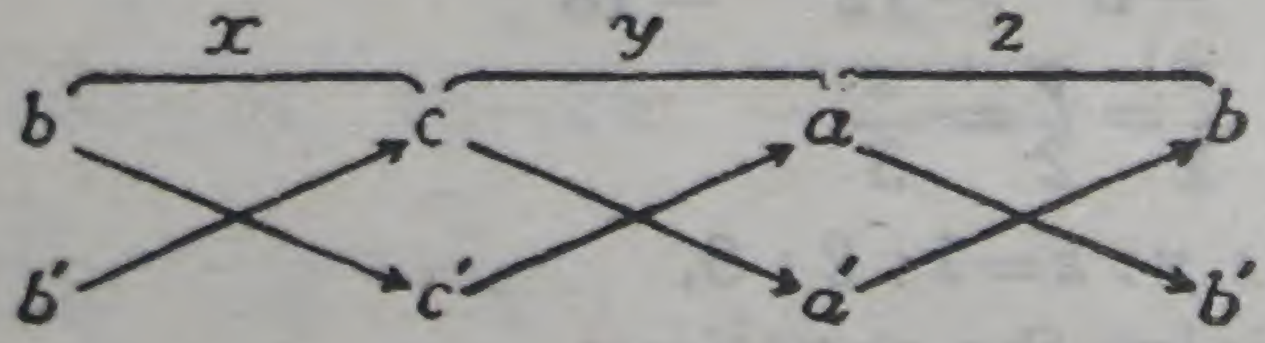
$$\frac{y}{ca'-c'a}=\frac{x}{bc'-b'c}$$

Similarly, $\dfrac{y}{ca'-c'a}=\dfrac{z}{ab'-a'b}$.

$$\therefore\quad \frac{x}{bc'-b'c}=\frac{y}{ca'-c'a}=\frac{z}{ab'-a'b} \quad \text{... ...} \quad \text{(v)}$$

This theorem is very useful. The rule of writing down the denominators under x, y and z may be thoroughly grasped.

Rule. **Write down in two rows, as given below, the co-efficients of y, z, x, y in order, from the original equations after they have been reduced to the standard**



form, and multiply the co-efficients diagonally as indicated by the arrows and put down

(i) $bc'-b'c$ **as the denominator under** x,
(ii) $ca'-c'a$ **as the denominator under** y,
(iii) $ab'-a'b$ **as the denominator under** z.

This is known as the rule of **Cross-multiplication.**

Cor. If $z=1$, the equations (i) and (ii) become

$$ax+by+c=0$$

and $$a'x+b'y+c'=0$$

and (v) becomes

$$\frac{x}{bc'-b'c}=\frac{y}{ca'-c'a}=\frac{1}{ab'-a'b}.$$

NOTE. The original theorem is useful in solving simultaneous equations of the first degree, involving three variables, provided two of them are of the form $ax+by+cz=0$, and the corollary gives us an elegant method of solving simultaneous equations of the first degree, involving two variables. The following examples are illustrative :

Example 1. Find $x:y:z$, if

$$4x-5y+2z=0 \quad \ldots \quad \ldots \quad \text{(i)}$$
$$2x-7y+4z=0 \quad \ldots \quad \ldots \quad \text{(ii)}$$

Arranging the co-efficients of y, z, x, y, we have

$$\begin{array}{cccc} -5 & 2 & 4 & -5 \\ -7 & 4 & 2 & -7, \end{array}$$

whence, by the rule of cross-multiplication, we obtain

$$\frac{x}{(-5)(4)-(-7)(2)}=\frac{y}{(2)(2)-(4)(4)}=\frac{z}{(4)(-7)-(2)(-5)}$$

or $$\frac{x}{-20+14}=\frac{y}{4-16}=\frac{z}{-28+10}$$

or $$\frac{x}{-6}=\frac{y}{-12}=\frac{z}{-18}$$

or $$\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$$

$\therefore$ $$x:y:z=1:2:3.$$

Example 2. Solve the equations

$$2x-3y+3z=0 \quad \ldots \quad \ldots \quad \text{(i)}$$
$$x-4y+5z=0 \quad \ldots \quad \ldots \quad \text{(ii)}$$
$$x+2y+3z=32 \quad \ldots \quad \ldots \quad \text{(iii)}$$

From (i) and (ii), by the rule of cross-multiplication,

$$\frac{x}{(-3)(5)-(-4)(3)}=\frac{y}{(3)(1)-(5)(2)}=\frac{z}{(2)(-4)-(1)(-3)}$$

or $$\frac{x}{-15+12}=\frac{y}{3-10}=\frac{z}{-8+3}$$

or $$\frac{x}{-3}=\frac{y}{-7}=\frac{z}{-5}$$

or $$\frac{x}{3}=\frac{y}{7}=\frac{z}{5}.$$

Let $$\frac{x}{3}=\frac{y}{7}=\frac{z}{5}=k.$$

$\therefore$ $$x=3k,\ y=7k,\ z=5k.$$

Substituting these values of x, y, z in (iii), we get

$$3k+14k+15k=32$$

or $$32k=32$$

or $$k=1$$

$\therefore$ $x=3,\quad y=7,\quad z=5.$

Example 3. Solve the equations

$$4x-5y+8=0 \quad \ldots \quad \ldots \quad \text{(i)}$$

$$2x-3y+6=0 \quad \ldots \quad \ldots \quad \text{(ii)}$$

From (i) and (ii), by rule of cross-multiplication,

$$\frac{x}{(-5)\times 6-(-3)\times 8}=\frac{y}{8\times 2-6\times 4}=\frac{1}{4\times(-3)-2\times(-5)}$$

or $$\frac{x}{-30+24}=\frac{y}{16-24}=\frac{1}{-12+10}$$

or $$\frac{x}{-6}=\frac{y}{-8}=\frac{1}{-2}$$

or $$\frac{x}{3}=\frac{y}{4}=1$$

$\therefore$ $x=3,\ y=4.$

EXERCISE 68.

Find $x : y : z$, if

1. $\left.\begin{aligned} x-2y+z &=0 \\ 9x-8y+3z &=0. \end{aligned}\right\}$

2. $\left.\begin{aligned} 6x-8y+7z &=0 \\ 2x-7y+11z &=0. \end{aligned}\right\}$

3. $\left.\begin{aligned} 4x-13y+8z &=0 \\ 7x+6y-9z &=0. \end{aligned}\right\}$

4. $\left.\begin{aligned} 2x-3y+4z &=0 \\ 7x+2y-6z &=0. \end{aligned}\right\}$

5. $\left.\begin{aligned} ax+by+cz &=0 \\ bx+cy+az &=0. \end{aligned}\right\}$

Solve the equations by the method of Cross-multiplication :

6. $\left.\begin{aligned} 4x-2y-z &=0 \\ 6x-y-4z &=0 \\ x+y+z &=25. \end{aligned}\right\}$

7. $\left.\begin{aligned} 5x+6y+8z &=0 \\ 3x+4y+6z &=0 \\ x+5y+16z &=3. \end{aligned}\right\}$

8. $\left.\begin{aligned} x+2y+z &=0 \\ 2x-5y-4z &=0 \\ 4x+5y+6z &=24. \end{aligned}\right\}$

9. $\left.\begin{aligned} 4(x+y) &=3(2z-y) \\ 5(x-2y) &=3(2y-3z) \\ 3(x-2)+5(y-3)+7(z-4) &=53. \end{aligned}\right\}$

10. $\left.\begin{aligned} 15x &=10y=6z \\ 3x+4y+5z &=172. \end{aligned}\right\}$

11. $\left.\begin{aligned} 4x-13y+8z &=0 \\ 7x+6y-9z &=0 \\ \frac{6}{x}+\frac{8}{y}+\frac{20}{z} &=8. \end{aligned}\right\}$

12. $\left.\begin{aligned} x-2y+z &=0 \\ 2x-5y+4z &=0 \\ 3x^2+4y^2+5z^2 &=48. \end{aligned}\right\}$

13. $\left.\begin{aligned} 7x-5y+z &=0 \\ 5x+2y-3z &=0 \\ 2x^3+5y^3-z^3 &=15. \end{aligned}\right\}$

14. $\left.\begin{aligned} x+y+z &=0 \\ ax+by+cz &=0 \\ \frac{x}{b-c}+\frac{y}{c-a}+\frac{z}{a-b} &=3. \end{aligned}\right\}$

15. $x-y=y-z=\dfrac{x+z}{6}=2.$

16. $\left.\begin{aligned} \frac{y+z}{4}=\frac{z+x}{3}=\frac{x+y}{2}. \\ x+y+z=27. \end{aligned}\right\}$

17. $\left.\begin{aligned} 3x+4y &=15 \\ 6y-5x &=-6. \end{aligned}\right\}$

18. $\left.\begin{aligned} 7y &=9x \\ 3y-4x+1 &=0. \end{aligned}\right\}$

19. $\left.\begin{aligned} 7x+4y&=5 \\ 5x+6y&=2. \end{aligned}\right\}$

20. $\dfrac{x+5}{y+2}=\dfrac{5x+3}{4y+1}=2.$

21. $\left.\begin{aligned} \frac{x+y}{2}+\frac{3x-5y}{4}&=2 \\ \frac{x}{14}+\frac{y}{18}&=1. \end{aligned}\right\}$

22. $\left.\begin{aligned} 3x+y-2p&=0 \\ -5x+py-3&=0. \end{aligned}\right\}$

23. $\left.\begin{aligned} lx+my&=p \\ mx&=ly. \end{aligned}\right\}$

24. $x-ay+b=ax-y+b=1.$

25. $\left.\begin{aligned} \frac{x}{a}+\frac{y}{b}&=c \\ \frac{x}{b}+\frac{y}{a}&=d. \end{aligned}\right\}$

26. If $x+5y=18$ and $3x+2y=41$, find the value of $x-8y$.

27. Find the values of x and y from the equation

$$(3x-2y+8)^2+(5x+13y-3)^2=0.$$

[*Hint.* As the sum of squares $=0$, each $=0$.]

Example 4. Solve the equations

$$\frac{3}{x}+\frac{4}{y}=18 \quad \ldots \quad \ldots \quad \ldots \quad \text{(i)}$$

$$\frac{2}{x}+\frac{5}{y}=19 \quad \ldots \quad \ldots \quad \ldots \quad \text{(ii)}$$

By transposition, we get

$$\frac{3}{x}+\frac{4}{y}-18=0 \quad \ldots \quad \ldots \quad \text{(iii)}$$

$$\frac{2}{x}+\frac{5}{y}-19=0 \quad \ldots \quad \ldots \quad \text{(iv)}$$

By the cross-multiplication method, we get

$$\frac{\frac{1}{x}}{(-76)-(-90)}=\frac{\frac{1}{y}}{(-36)-(-57)}=\frac{1}{15-8}$$

or $\dfrac{\frac{1}{x}}{14}=\dfrac{\frac{1}{y}}{21}=\dfrac{1}{7}$

$\therefore \dfrac{1}{x}=\dfrac{14}{7}=2$ and $\dfrac{1}{y}=\dfrac{21}{7}=3$

$\therefore x=\dfrac{1}{2}$ and $y=\dfrac{1}{3}$.

Example 5. Solve the equations

$$5x+4y=9xy \quad \ldots \quad \ldots \text{ (i)}$$
$$3x+2y=5xy \quad \ldots \quad \ldots \text{ (ii)}$$

Dividing (i) and (ii) by xy, we get

$$\left.\begin{aligned}\frac{5}{y}+\frac{4}{x}&=9\\ \frac{3}{y}+\frac{2}{x}&=5\end{aligned}\right\}$$

or
$$\frac{4}{x}+\frac{5}{y}-9=0 \quad \ldots \quad \ldots \text{ (iii)}$$
$$\frac{2}{x}+\frac{3}{y}-5=0 \quad \ldots \quad \ldots \text{ (iv)}$$

By the cross-multiplication method, we get

$$\frac{\frac{1}{x}}{-25+27}=\frac{\frac{1}{y}}{-18+20}=\frac{1}{12-10}$$

or
$$\frac{\frac{1}{x}}{2}=\frac{\frac{1}{y}}{2}=\frac{1}{2}$$

$$\therefore \quad \frac{1}{x}=\frac{1}{y}=1$$

$$\therefore \quad x=1 \text{ and } y=1.$$

Solve the equations by the method of cross-multiplication :

28. $\left.\begin{aligned}\frac{3}{x}+\frac{4}{y}&=18\\ \frac{2}{x}+\frac{5}{y}&=19.\end{aligned}\right\}$

29. $\left.\begin{aligned}\frac{3}{x}-\frac{2}{y}&=-4\\ \frac{2}{x}+\frac{3}{y}&=19.\end{aligned}\right\}$

30. $\left.\begin{aligned}\frac{3}{x}-\frac{4}{y}&=5\\ \frac{4}{x}-\frac{5}{y}&=6.\end{aligned}\right\}$

31. $\left.\begin{aligned}\frac{3}{x}+\frac{5}{y}&=11\\ \frac{2}{x}-\frac{3}{y}&=-5\tfrac{1}{13}.\end{aligned}\right\}$

32. $\left.\begin{aligned}\frac{1}{3x}+\frac{1}{5y}&=2\\ \frac{1}{4x}+\frac{1}{6y}&=\frac{19}{12}.\end{aligned}\right\}$

33. $\left.\begin{aligned}2x+\frac{3}{y}&=5\\ 5x-\frac{2}{y}&=3.\end{aligned}\right\}$

34. $\left.\begin{aligned}\frac{2}{3x}-\frac{5}{4y}&=\frac{1}{12}\\ \frac{3}{4x}-\frac{7}{6y}&=\frac{1}{3}.\end{aligned}\right\}$

35. $\left.\begin{aligned}\frac{1}{5x}+\frac{y}{9}&=5\\ \frac{1}{3x}+\frac{y}{2}&=14.\end{aligned}\right\}$

36. $\left.\begin{aligned}\frac{a}{x}+\frac{b}{y}&=m\\ \frac{b}{x}+\frac{a}{y}&=n.\end{aligned}\right\}$

37. $\left.\begin{aligned}\frac{2}{ax}+\frac{3}{by}&=5\\ \frac{5}{ax}-\frac{2}{by}&=3.\end{aligned}\right\}$

38. $\left.\begin{aligned}\frac{8}{x+y}+\frac{6}{x-y}&=25\\ \frac{4}{x+y}-\frac{3}{x-y}&=10.\end{aligned}\right\}$

39. $\left.\begin{aligned}10y+14x&=9xy\\ 4x-6y&=7xy.\end{aligned}\right\}$

40. $\left.\begin{aligned}2y-x&=4xy\\ \frac{4}{y}-\frac{3}{x}&=9.\end{aligned}\right\}$

41. $\frac{2+5x}{3x}=\frac{y+4}{2}=\frac{2xy+9x+2}{6x}.$

2. Simultaneous equations with three variables.

Example 1. Solve the equations

$$4x+2y-3z=2 \quad \ldots \quad \text{(i)}$$
$$3x+4y-2z=10 \quad \ldots \quad \text{(ii)}$$
$$2x-5y+4z=5 \quad \ldots \quad \text{(iii)}$$

Multiplying (i) by 5 and then subtracting (ii) from it,

$$17x+6y-13z=0 \quad \ldots \quad \text{(iv)}$$

Multiplying (iii) by 2 and then subtracting it from (ii),

$$-x+14y-10z=0 \quad \ldots \quad \text{(v)}$$

By the cross-multiplication method from (iv) and (v), we get

$$\frac{x}{-60+182}=\frac{y}{13+170}=\frac{z}{238+6}$$

or $$\frac{x}{122}=\frac{y}{183}=\frac{z}{244}$$

or $$\frac{x}{2}=\frac{y}{3}=\frac{z}{4}=k \text{ (Let each be equal to } k.)$$

$\therefore$ $x=2k$, $y=3k$ and $z=4k$.

Substituting these values in (i), we get

$$8k+6k-12k=2$$

or $$k=1$$

$\therefore$ $x=2$, $y=3$, $z=4$.

Example 2. Solve the equations

$$x+5y+6z=3 \quad \ldots \quad \ldots \quad \text{(i)}$$
$$2x-3y+7z=22 \quad \ldots \quad \ldots \quad \text{(ii)}$$
$$5x+y+8z=19 \quad \ldots \quad \ldots \quad \text{(iii)}$$

Multiplying (i) by 2 and then subtracting (ii) from it,

$$13y+5z=-16$$

or $$13y+5z+16=0 \quad \ldots \quad \ldots \quad \text{(iv)}$$

Multiplying (i) by 5 and then subtracting (iii) from it,

$$24y+22z=-4$$

or $$12y+11z+2=0 \quad \ldots \quad \ldots \quad \text{(v)}$$

From (iv) and (v), by the method of cross-multiplication,

$$\frac{y}{10-176}=\frac{z}{192-26}=\frac{1}{143-60}$$

or $$\frac{y}{-166}=\frac{z}{166}=\frac{1}{83}$$

$\therefore \quad y=-2, \ z=2.$

Substituting these values in (i), we get $x=1$.

EXERCISE 69.

Solve the equations:

1. $\left.\begin{aligned} x+y+z&=1\\ 2x+3y+z&=4\\ 4x+9y+z&=16. \end{aligned}\right\}$

2. $\left.\begin{aligned} 4x-3y+2z&=8\\ 3x-4y+5z&=6\\ -6x+5y+7z&=-1. \end{aligned}\right\}$

3. $\left.\begin{aligned} 3x-2y+4z&=5\\ 5x-7y+8z&=8\\ 11x-10y+16z&=21. \end{aligned}\right\}$

4. $\left.\begin{aligned} x+5y-4z&=5\\ 3x-2y+2z&=14\\ -10x+8y+z&=6. \end{aligned}\right\}$

5. $x+2y+3z=3x+y+2z=2x+3y+z=6.$

***6.** $\left.\begin{aligned} \frac{1}{x}+\frac{1}{y}+\frac{1}{z}&=6\\ \frac{2}{x}-\frac{3}{y}+\frac{4}{z}&=8\\ \frac{3}{x}-\frac{4}{y}+\frac{5}{z}&=10. \end{aligned}\right\}$

***7.** $\left.\begin{aligned} 3x+\frac{4}{y}+\frac{5}{z}&=-13\\ 4x+\frac{5}{y}+\frac{3}{z}&=7.\\ 5x+\frac{3}{y}+\frac{4}{z}&=6. \end{aligned}\right\}$

***8.** $\left.\begin{aligned} x-z&=1\\ 2y+z&=11\\ 2x+3y&=23. \end{aligned}\right\}$

SECTIONAL REVISION IV

TEST PAPERS

PAPER 1

1. Explain the terms :
   (i) duplicate ratio and sub-duplicate ratio,
   (ii) triplicate ratio and sub-triplicate ratio.

2. Prove that a ratio of greater inequality is increased and a ratio of less inequality is diminished by subtracting the same positive quantity from both the terms.

3. If $\frac{6a-5b}{6a+5b} = \frac{6c-5d}{6c+5d}$, shew that $\frac{a}{b} = \frac{c}{d}$.

4. If $\frac{a}{b} = \frac{c}{d} = \frac{e}{f}$, shew that each is equal to $\frac{a+c+e}{b+d+f}$.

5. Solve the equations :

   (i) $\frac{a}{ax+1} + \frac{b}{bx+1} = \frac{2c}{cx+1}$.

   (ii) $9\left(\frac{x-a}{x+a}\right)^3 = \frac{x-a}{x+a}$.

6. The co-ordinates of P and Q are $(-7, 5)$ and $(+8, -4)$ respectively. Find the length of PQ correct to 2 places of decimals.

7. Solve graphically the equations

$$\left.\begin{aligned} x+y&=1 \\ x-y&=5. \end{aligned}\right\}$$

PAPER 2

1. Find the compounded ratio of the following three ratios :

   $3a : 4b$, $b^2 : 5ac$ and $5c^2 : 6ab$.

2. If $6+x : 8+x$ be the sub-duplicate ratio of $4 : 9$, find x.

3. If $a : b = c : d$, then $\frac{4a+3b}{4a-3b} = \frac{4c+3d}{4c-3d}$.

4. If $\frac{x}{a+b-c}=\frac{y}{b+c-a}=\frac{z}{c+a-b}$, shew that each of these fractions $=\frac{x+y+z}{a+b+c}$.

5. Solve the equations

(i) $(x^2+5x-4)(x^2+3x-2)=(x^2-5x+4)(x^2-3x+2)$.

(ii) $\frac{6x+5}{8x+3}=\frac{3x+2}{4x+1}$.

6. Find the equation of a straight line passing through (3, 0) and (−3, 4).

PAPER 3

1. What must be added to the terms of 5 : 8 to make it equal to 9 : 10?

2. If $x : y=5 : 6$, find the value of $\frac{3x+4y}{4x+3y}$.

3. Two vessels contain a mixture of wine and water in the ratio of 5 : 3 and 3 : 1. In what ratio must liquids be drawn from each to give a mixture of wine and water in the ratio of 2 : 1?

4. If $x=\frac{ab}{a+b}$, find the value of $\frac{x+a}{x-a}+\frac{x+b}{x-b}$.

5. Solve the equations:

(i) $\frac{5-6x}{3+4x}=\frac{2+3x}{5-2x}$.

(ii) $\frac{(x+2)(x+3)}{(x+4)(x+5)}=\frac{x+5}{x+9}$.

6. Find the equation of a straight line which is parallel to $5x+4y=3$ and passes through $(6, -4)$.

7. Solve the following equations by the method of cross-multiplication:

$$\left.\begin{aligned}\frac{3}{x}+\frac{2}{y}-2&=0\\ \frac{4}{x}+\frac{3}{y}-2\tfrac{5}{6}&=0.\end{aligned}\right\}$$

PAPER 4

1. If $7x-6y=5x+2y$, find the value of $x : y$.

2. The bases of two triangles are in the ratio of 5 : 6 and their altitudes are in the ratio of 8 : 9. Find the ratio between their areas.

3. If $a : b :: c : d$, prove that $\frac{a+b}{a-b}=\frac{c+d}{c-d}$.

4. If l, m, n are in continued proportion and if $l(m-n)=2m$, prove that $l-n=2\left(\frac{l+m}{l}\right)$.

5. Solve the equations:

(i) $\left(\frac{2x-3}{2x-5}\right)^2=\frac{x-3}{x-5}$.

(ii) $(x-3)(x+4)(x+7)=(x+2)(x+1)(x+5$

6. Draw the graph of the function $\frac{2x-3}{6}$.

7. Solve the equations:
$$\left.\begin{aligned}\frac{3}{x}+\frac{2}{y}&=\frac{17}{12}\\ \frac{2}{z}-\frac{1}{y}&=\frac{2}{3}\\ \frac{1}{x}+\frac{1}{z}&=\frac{3}{4}.\end{aligned}\right\}$$

PAPER 5

1. Prove that $\frac{0}{n}=0, \frac{m}{\infty}=0$ and $\frac{m}{0}=\infty$, where m and n are finite quantities.

2. What must be added to each of the numbers 12, 16, 21 so that the resulting numbers may be in continued proportion?

3. If $\frac{2a^2+3ab+4b^2}{5a^2+6ab+7b^2}=\frac{2c^2+3cd+4d^2}{5c^2+6cd+7d^2}$, prove that $a : b=c : d$.

4. If $\frac{a}{y+z-x}=\frac{b}{z+x-y}=\frac{c}{x+y-z}$, shew that
$$\frac{x}{b+c}=\frac{y}{c+a}=\frac{z}{a+b}.$$

5. Solve the equations:

(i) $81\left(\frac{a-x}{a+x}\right)^3=\frac{a-x}{a+x}$.

(ii) $\frac{x^2-5x+7}{5x-7}=\frac{x^2-3x+5}{3x-5}$.

6. Find the equations of the sides of a triangle whose vertices are $(-5, 4)$, $(6,-2)$ and $(3, 2)$.

PAPER 6

1. Prove that $\frac{0}{0}$ is indeterminate.

2. Find two numbers of the same two digits which are in the ratio of 8 : 3.

3. If $\frac{pa+qb}{pc+qd}=\frac{b^2c}{d^2a}$, prove that $a : b=c : d$.

4. If $\frac{a}{x}=\frac{b}{y}=\frac{c}{z}$, shew that $\frac{abc}{xyz}=\frac{a^3+b^3+c^3}{x^3+y^3+z^3}$.

5. Solve the equations:

(i) $\frac{x^2-3x+4}{2x^2-5x+1}=\frac{x^2+3x-4}{2x^2+5x-1}$.

(ii) $\frac{x^3+3x}{3x^2+1}=\frac{91}{37}$.

6. Solve graphically the equations $\left.\begin{aligned} y&=2x-1\\ 6y&=2x-3.\end{aligned}\right\}$

7. The average weights of infants for the first 12 months are given below in lbs.:

Months	1	2	3	4	5	6	7	8	9	10	11	12
Weights	8	$9\frac{1}{2}$	11	$12\frac{1}{2}$	14	$15\frac{1}{2}$	$16\frac{3}{4}$	18	19	20	21	22

Draw the graph and compare the weight of any infant you know.

PAPER 7

Find the value of $\frac{x^2+2x-35}{x^2+4x-21}$ when $x=-7$.

If $1+x$, $2+x$, $4+x$ are in continued proportion, find x.

3. If $a : b = c : d$, prove that $(6a-5c)(6b+5d) = (6a+5c)(6b-5d)$.

4. If $\frac{x}{a} = \frac{y}{b}$, prove that $(x^2+y^2)(a^2+b^2) = (ax+by)^2$.

5. Solve the equations :

(i) $\frac{x^2+x-30}{x^2-25} = 2$.

(ii) $\frac{3x-2}{x-1} + \frac{6x-13}{2x-5} = \frac{3x-5}{x-2} + \frac{6x-7}{2x-3}$.

6. Solve graphically and verify the solution of
$$\left.\begin{aligned} \frac{x+y}{3} &= 5 \\ 2x-3y &= 5. \end{aligned}\right\}$$

PAPER 8

1. The soldiers in two armies when met in a battle were in the ratio of 5 : 3, their respective losses were in the ratio of 3 : 2 and their survivors as 69 : 41. If the number of survivors in the smaller army be 16,400, find the original number of soldiers.

2. Find (i) the mean proportional between 16, 49 ;
(ii) the fourth proportional to 12, 15, 18 ;
and (iii) the third proportional to 24, 30.

3. If $2p = m+n$ and $\frac{2}{p} = \frac{1}{x} + \frac{1}{y}$, shew that $m : x = n : y$.

4. If a, b, c, d are in continued proportion, shew that $ab+bc+cd$ is the mean proportional between $a^2+b^2+c^2$ and $b^2+c^2+d^2$.

5. Solve the equations :

(i) $\frac{2x+7}{x+2} + \frac{3x+13}{x+3} = \frac{5x+27}{x+4}$.

(ii) $\frac{x-8}{x-10} - \frac{x-5}{x-7} = \frac{x-7}{x-9} - \frac{x-4}{x-6}$.

6. Find the speed of a train if 90 minutes are saved in 150 miles when the speed is increased by 5 miles an hour.

7. Find the vertices of the triangle formed by the straight lines
$$\left.\begin{aligned} 8x+5y+7 &= 0 \\ 3x+7y-23 &= 0 \\ 5x-2y-11 &= 0. \end{aligned}\right\}$$

CHAPTER XVI

INDICES

1. In a^5, ^{5} is the index and it indicates that a is repeated as a factor 5 times, or $a^5 = a \times a \times a \times a \times a$.

This definition of an index does not obviously involve the idea of direction. An index used hitherto is neither a positive integer, nor a negative one—*it* is a non-directed integer. Thus a^{+5} or a^{-5} has no meaning. Again, as an index indicates the number of times a factor is repeated, $a^{\frac{1}{5}}$ or $a^{\frac{2}{3}}$ is also meaningless.

In order to include the directed and fractional indices in the domain of Algebra, we have to extend the meaning of an index.

We have already established the following laws of indices from the ordinary meaning of an index :

$$a^m \times a^n = a^{m+n} \quad \ldots \quad \ldots \quad \ldots \quad \text{(I)}$$

$$a^m \div a^n = a^{m-n} \quad \ldots \quad \ldots \quad \ldots \quad \text{(II)}$$

$$(a^m)^n = a^{mn} \quad \ldots \quad \ldots \quad \ldots \quad \text{(III)}$$

Now the problem is to put such interpretation on a^{+m}, a^{-n}, $a^{\frac{m}{n}}$ and a^0 that they may be included in the domain of Algebra.

The extension of Algebraic domain, with a view to include such *mysterious* and *foreign elements*, is possible only if these elements would obey all the rules which constitute the Algebraic government.

Starting with the assumption that these foreign and mysterious elements do accept all the rules of Algebraic government, we lay down the following principles :

(i) that a^{+m} means exactly the same thing as a^m ;

(ii) that $a^{\frac{m}{n}}$, a^0, a^{-n} are subject to the ordinary laws of Algebra.

2. Meaning of $a^{\frac{m}{n}}$.

By the law of indices,

$$a^{\frac{2}{3}} \times a^{\frac{2}{3}} \times a^{\frac{2}{3}} = a^{\frac{2}{3}+\frac{2}{3}+\frac{2}{3}}$$

$$\therefore \quad \left(a^{\frac{2}{3}}\right)^3 = a^{\frac{2}{3}\times 3} = a^2.$$

Taking the cube root of both sides, we have

$$a^{\frac{2}{3}} = \sqrt[3]{a^2}$$

Similarly, $a^{\frac{4}{5}} = \sqrt[5]{a^4}$ and $a^{\frac{7}{11}} = \sqrt[11]{a^7}$.

For further generalisation, we proceed as below.

By the law of indices,

$$a^{\frac{m}{n}} \times a^{\frac{m}{n}} \times a^{\frac{m}{n}} \times \ldots \text{ to } n \text{ factors}$$

$$a^{\frac{m}{n}+\frac{m}{n}+\frac{m}{n}+\ldots \text{to } n \text{ terms}}$$

$$\therefore \quad \left(a^{\frac{m}{n}}\right)^n = a^{\frac{m}{n}\times n} = a^m.$$

Taking the nth root of both sides, we have

$$a^{\frac{m}{n}} = \sqrt[n]{a^m}.$$

Hence $a^{\frac{m}{n}}$ is the nth root of a^m, and in future we shall always take $a^{\frac{m}{n}} = \sqrt[n]{a^m}$

Cor. $a^{\frac{1}{n}} = \sqrt[n]{a^1} = \sqrt[n]{a}.$

Thus, $a^{\frac{1}{2}} = \sqrt{a}$, $a^{\frac{1}{3}} = \sqrt[3]{a}$, $a^{\frac{1}{4}} = \sqrt[4]{a}$, and so on.

Since $a^{\frac{1}{n}} \times a^{\frac{1}{n}} \times a^{\frac{1}{n}} \times \ldots\ldots m \text{ factors} = a^{\frac{1}{n}\times m} = a^{\frac{m}{n}}$

and $a^{\frac{1}{n}} = \sqrt[n]{a}$,

$$\left(\sqrt[n]{a}\right)^m = a^{\frac{m}{n}}, \text{ or } a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^m$$

Thus, $a^{\frac{m}{n}}$ may be interpreted either as the nth root of the mth power of a, or as the mth power of the nth root of a.

3. Meaning of a^0.

By the law of indices,

$$a^0 \times a^m = a^{0+m} = a^m.$$

Dividing both sides by a^m, we have

$$\frac{a^0 \times a^m}{a^m} = \frac{a^m}{a^m}$$

$$\therefore \quad a^0 = 1.$$

Hence for all finite values of a, $a^0 = 1$.

4. Meaning of a^{-m}.

By the law of indices,

$$a^m \times a^{-m} = a^{m-m} = a^0 = 1.$$

Dividing both sides by a^m, we have

$$a^{-m} = \frac{1}{a^m}$$

Also $$a^m = \frac{1}{a^{-m}}.$$

Cor. $$\frac{a^{-m}}{b^{-n}} = a^{-m} \times \frac{1}{b^{-n}} = \frac{1}{a^m} \times b^n = \frac{b^n}{a^m}.$$

Hence any quantity with a *negative* index denotes the **reciprocal** of the same with the same *positive* index and *vice-versa*.

Example 1. Find the value of $27^{\frac{4}{3}}$.

$$27^{\frac{4}{3}} = \left(\sqrt[3]{27}\right)^4 = 3^4 = 81.$$

Example 2. Find the value of $9^{-\frac{3}{2}}$.

$$9^{-\frac{3}{2}} = \frac{1}{9^{\frac{3}{2}}} = \frac{1}{(\sqrt{9})^3}$$

$$= \frac{1}{3^3} = \frac{1}{27}.$$

Example 3. Multiply together $\sqrt[4]{a^3}$, $a^{\frac{3}{2}}$, $\frac{1}{a^{-4}}$ and $a^{-\frac{1}{4}}$.

$$\text{The required product} = a^{\frac{3}{4}} \times a^{\frac{3}{2}} \times a^4 \times a^{-\frac{1}{4}}$$
$$= a^{\frac{3}{4}+\frac{3}{2}+4-\frac{1}{4}}$$
$$= a^6.$$

Example 4. Express $\sqrt[4]{a^{-3}} \div (\sqrt[6]{a})^{-5}$ without radical signs and negative indices.

$$\sqrt[4]{a^{-3}} \div \left(\sqrt[6]{a}\right)^{-5} = a^{-\frac{3}{4}} \div a^{-\frac{5}{6}}$$
$$= \frac{1}{a^{\frac{3}{4}}} \div \frac{1}{a^{\frac{5}{6}}}$$
$$= \frac{a^{\frac{5}{6}}}{a^{\frac{3}{4}}} = a^{\frac{5}{6}-\frac{3}{4}}$$
$$= a^{\frac{1}{12}}.$$

EXERCISE 70.

Express the following without fractional and negative indices:

1. $a^{\frac{3}{5}}$.
2. $x^{-\frac{3}{5}}$.
3. $x^{-\frac{2}{3}} \times 6a^{-\frac{1}{2}}$.
4. $9p^{-2} \times p^{-\frac{3}{4}}$.
5. $m^{-\frac{2}{3}} \div a^{-\frac{3}{2}}$.
6. $\frac{4}{x^{-\frac{2}{3}}}$.
7. $\sqrt[5]{m^3} \div \sqrt[5]{x^{-2}}$.
8. $\sqrt[2m]{x^{-7}} \times \sqrt{a^5}$.
9. $\sqrt[4m]{x^{12}} \div \sqrt[2m]{x^{-5}}$.

Express the following without radical signs and negative indices:

10. $(\sqrt[5]{x})^7$.
11. $(\sqrt[6]{m})^{-9}$.
12. $\frac{1}{\sqrt[3]{m^{-2}}}$.
13. $\frac{1}{(\sqrt[3]{a})^{-4}}$.
14. $\sqrt[4]{x^6} \div (\sqrt[6]{x})^{-1}$.
15. $\sqrt[4]{m^{-5}} \div (\sqrt[8]{m})^{-6}$.

Find the value of :

16. $4^{-\frac{1}{2}}$. 17. $8^{\frac{2}{3}}$. 18. $16^{\frac{3}{4}}$. 19. $64^{\frac{2}{3}}$.

20. $32^{-\frac{3}{5}}$. 21. $27^{-\frac{2}{3}}$. 22. $125^{\frac{2}{3}}$. 23. $81^{-\frac{3}{4}}$.

24. $128^{-\frac{3}{7}}$. 25. $343^{-\frac{2}{3}}$. 26. $\sqrt[5]{243^2}$ 27. $\frac{1}{9^{-2}}$.

28. $\left(\frac{1}{216}\right)^{-\frac{2}{3}}$. 29. $\frac{1}{\sqrt{16^{\frac{3}{2}}}}$. 30. $\sqrt[3]{(16^{-\frac{3}{4}})(125^{-2})}$

Simplify :

31. $16^{\frac{1}{2}}+49^{\frac{1}{2}}$. 32. $\frac{1}{8^{-\frac{1}{3}}}+\frac{1}{27^{-\frac{1}{3}}}$ 33. $\frac{(2\cdot5)^3}{10}$

34. $\frac{x+x^{-1}}{x-1}$. 35. $\frac{a^{2m+n}\times a^{3m-5n}}{a^{4m-3n}}$.

5. To prove that $(a^m)^n=a^{mn}$ for all values of m &

Case I. When n is a positive integer.

$$(a^m)^n=a^m\times a^m\times a^m\ldots\ldots\text{to } n \text{ factors}$$
$$=a^{m+m+m+\ldots \text{ to } n \text{ terms}}=a^{mn}.$$

Case II. When n is a positive fraction.

Let $n=\frac{p}{q}$, where p and q are positive integers.

Now $$(a^m)^n=(a^m)^{\frac{p}{q}}=\sqrt[q]{(a^m)^p}$$
$$=\sqrt[q]{a^{mp}} \quad =a^{\frac{mp}{q}}$$
$$=a^{mn}.$$

Case III. When n is any negative quantity.

Let $n=-p$, then p is a positive quantity.

Now $$(a^m)^n=(a^m)^{-p}=\frac{1}{(a^m)^p}$$
$$=\frac{1}{a^{mp}}=a^{-mp}$$
$$=a^{mn}.$$

Example 1. $(9^{\frac{2}{3}})^{\frac{3}{4}}=9^{\frac{2}{3}\times\frac{3}{4}}=9^{\frac{1}{2}}=\sqrt{9}=3.$

Example 2. $(27^{-\frac{2}{3}})^{\frac{1}{2}} = 27^{-\frac{2}{3}\times\frac{1}{2}} = 27^{-\frac{1}{3}} = \frac{1}{27^{\frac{1}{3}}} = \frac{1}{\sqrt[3]{27}} = \frac{1}{3}$.

6. To prove that $a^n b^n = (ab)^n$ for all values of n.

Case I. When n is a positive integer.

$$a^n b^n = (a \times a \times a \ldots \text{to } n \text{ factors}) \times (b \times b \times b \ldots \text{to } n \text{ factors})$$
$$= (ab \times ab \times ab \ldots \text{to } n \text{ factors})$$
$$= (ab)^n.$$

Case II. When n is a positive fraction.

Let $n = \frac{p}{q}$, where p and q are both positive integers, then $p = nq$.

Now $(ab)^n = (ab)^{\frac{p}{q}} = \sqrt[q]{(ab)^p}$

$$= \sqrt[q]{a^p b^p} = \sqrt[q]{a^{nq} b^{nq}}$$
$$= \sqrt[q]{(a^n)^q (b^n)^q} = \sqrt[q]{(a^n b^n)^q}$$
$$= (a^n b^n)^{\frac{q}{q}} = a^n b^n.$$

Case III. When n is any negative quantity.

Let $n = -p$, then p is a positive quantity.

Now $(ab)^n = (ab)^{-p} = \frac{1}{(ab)^p}$

$$= \frac{1}{a^p b^p} = a^{-p} b^{-p} = a^n b^n.$$

Hence generally $(abc \ldots)^n = a^n \times b^n \times c^n \ldots$. for all values of n.

Cor. $\frac{a^n}{b^n} = a^n b^{-n} = a^n (b^{-1})^n = (ab^{-1})^n = \left(\frac{a}{b}\right)^n$.

Example 3. $(8^{-2} \times 9^{\frac{3}{4}})^{-\frac{2}{3}} = (8^{-2})^{-\frac{2}{3}} \times (9^{\frac{3}{4}})^{-\frac{2}{3}}$

$$= 8^{\frac{4}{3}} \times 9^{-\frac{1}{2}} = (\sqrt[3]{8})^4 \times \frac{1}{\sqrt{9}}$$
$$= 2^4 \times \frac{1}{3} = \frac{16}{3}.$$

Example 4. $\sqrt[3]{a^{-4}b^2} \times \sqrt{a^3b} \times \sqrt[6]{ab^{-3}}$

$$= a^{-\frac{4}{3}} b^{\frac{2}{3}} \times a^{\frac{3}{2}} b^{\frac{1}{2}} \times a^{\frac{1}{6}} b^{-\frac{1}{2}}$$

$$= a^{-\frac{4}{3}+\frac{3}{2}+\frac{1}{6}} \times b^{\frac{2}{3}+\frac{1}{2}-\frac{1}{2}}$$

$$= a^{\frac{1}{3}} \times b^{\frac{2}{3}}$$

$$= \sqrt[3]{ab^2}.$$

EXERCISE 71.

Simplify :

1. $(x^{-\frac{2}{3}})^6$.
2. $(x^{-\frac{3}{4}}y^{\frac{5}{8}})^{\frac{3}{4}}$.
3. $(x^{-\frac{1}{3}}b^{-2})^{-3}$.
4. $(x^6 y^{\frac{5}{4}})^{-\frac{4}{3}}$.
5. $(\sqrt[4]{m^3n^4})^8$.
6. $(\sqrt[8]{a^{12}b^{-8}})^{-4}$
7. $\sqrt[4]{m^6 \quad \sqrt[4]{m^{-3}}}$
8. $\sqrt{x^{-6}y^8} \times \sqrt[4]{x^4y^{-12}}$.
9. $\sqrt{a^{-1}\sqrt{a^3\sqrt{a^{-4}}}}$.
10. $\sqrt[4]{a^{-4}\sqrt{b^5}} \times \sqrt{a\sqrt[4]{b^3}}$.
11. $(27a^3 \div 64b^{-3})^{\frac{2}{3}}$
12. $(125a^3 \div 27b^{-3})^{-\frac{2}{3}}$.
13. $\{(a^{\frac{1}{3}}b^{-\frac{2}{3}})^{-6} \times (a^{-\frac{1}{3}}b^{\frac{2}{3}})^{-6}\}^{\frac{1}{4}}$
14. $\sqrt{xy^{-2}z^3} \div \sqrt[3]{(x^3y^2z^{-3})}^{-1}$
15. $\left\{(x^{a+b-c} \times x^{a-b+c})^b\right\}^c$.
16. $\left(\frac{x^m}{x^n}\right)^{m+n} \times \left(\frac{x^n}{x^l}\right)^{n+l} \times \left(\frac{x^l}{x^m}\right)^{l+m}$
17. Which is greater $(2^3)^2$ or 2^{3^2}, and by how much ?

7. Algebraical operations involving fractional and negative indices.

As in the processes of multiplication, division, factorisation and square root, it is necessary to arrange the terms in ascending or descending powers of some common letter, the student should have sufficient practice in such arrangements while dealing with expressions involving fractional and negative indices.

Example 1. Arrange the following expressions in ascending powers of x:

(i) $x+x^0+x^{-1}+x^{-2}+x^2$.

(ii) $3x^{\frac{1}{2}}+x^{-1}+4x^{-\frac{1}{2}}+x^{\frac{1}{4}}+x^0$.

(iii) $x^{\frac{1}{8}}-2x^{-\frac{3}{4}}+3x^{-\frac{1}{4}}+1-7x^2+5x$.

(i) Since $0>-1$ and $-1>-2$, the proper arrangement of the terms in the first expression would be

$$x^{-2}+x^{-1}+x^0+x+x^2.$$

(ii) Since $0>-\frac{1}{2}$ and $-\frac{1}{2}>-1$, the proper arrangement of the terms in the second expression would be

$$x^{-1}+4x^{-\frac{1}{2}}+x^0+x^{\frac{1}{4}}+3x^{\frac{1}{2}}.$$

(iii) Since $1=x^0$ and $0>-\frac{1}{4}$ and $-\frac{1}{4}>-\frac{3}{4}$, the proper arrangement of the terms in the third expression would be

$$-2x^{-\frac{3}{4}}+3x^{-\frac{1}{4}}+1+x^{\frac{1}{8}}+5x-7x^2.$$

Ex. Arrange the above three expressions in descending powers of x.

Example 2. Multiply $ab^{-1}+1+a^2b^{-2}$ by $a^2b^{-2}+1-ab^{-1}$.

Arranging both the expressions in ascending powers of a and proceeding as usual, we have

$$\begin{array}{r} 1+ab^{-1}+a^2b^{-2} \\ 1-ab^{-1}+a^2b^{-2} \\ \hline 1+ab^{-1}+a^2b^{-2} \\ -ab^{-1}-a^2b^{-2}-a^3b^{-3} \\ \quad +a^2b^{-2}+a^3b^{-3}+a^4b^{-4} \\ \hline 1 \qquad +a^2b^{-2} \qquad +a^4b^{-4}. \end{array}$$

Example 3. Divide $11x^{\frac{1}{5}}-6+x^{\frac{3}{5}}-6x^{\frac{2}{5}}$ by $x^{\frac{1}{5}}-3$.

Arranging the dividend and divisor according to descending powers of x and proceeding as usual, we have

$$x^{\frac{1}{5}}-3)x^{\frac{3}{5}}-6x^{\frac{2}{5}}+11x^{\frac{1}{5}}-6(x^{\frac{2}{5}}-3x^{\frac{1}{5}}+2$$
$$x^{\frac{3}{5}}-3x^{\frac{2}{5}}$$
$$-3x^{\frac{2}{5}}+11x^{\frac{1}{5}}$$
$$-3x^{\frac{2}{5}}+9x^{\frac{1}{5}}$$
$$2x^{\frac{1}{5}}-6$$
$$2x^{\frac{1}{5}}-6$$

EXERCISE 72.

1. Arrange the following expressions according to ascending powers of x:

(i) $4x^2+1+3x^{-1}+5x^{-2}-7x$.

(ii) $3+6x^{-3}+5x^3+4x^{-2}+7x^{-1}$.

(iii) $x+4+6x^{-\frac{1}{2n}}+5x^{\frac{1}{n}}+3x^{\frac{1}{2n}}-x^{-\frac{1}{n}}$, where n is a positive integer.

(iv) $7x^{\frac{1}{n}}-2+x+6x^{-\frac{1}{3n}}+5x^{\frac{1}{2n}}$, where n is a positive integer.

2. Arrange the following expressions according to descending powers of y:

(i) $2+x^{-2}y^2+x^2y^{-2}$.

(ii) $x^{\frac{2}{3}}y^{-\frac{2}{3}}-1+x^{-\frac{2}{3}}y^{\frac{2}{3}}$.

(iii) $3y^{-3}-xy^{-2}+1-5x^2y^{-1}+7x^3$.

Multiply:

3. $a^{\frac{2}{3}}-1+a^{-\frac{2}{3}}$ by $a^{\frac{1}{3}}+a^{-\frac{1}{3}}$.

4. $a-a^{\frac{1}{2}}b^{\frac{1}{2}}+b$ by $a^{\frac{1}{2}}+b^{\frac{1}{2}}$.

5. $a+3b^{\frac{1}{2}}+4c^{\frac{1}{3}}$ by $a-3b^{\frac{1}{2}}+4c^{\frac{1}{3}}$.

6. $a^{\frac{2}{3}}+2a^{\frac{1}{2}}+3a^{\frac{1}{3}}+2a^{\frac{1}{6}}+1$ by $1+a^{\frac{1}{3}}-2a^{\frac{1}{6}}$.

7. $x^{-1}+y^{-1}+x^{-\frac{1}{2}}y^{-\frac{1}{2}}$ by $x^{-1}-x^{-\frac{1}{2}}y^{-\frac{1}{2}}+y^{-1}$.

8. $a^{\frac{2}{3}}+b^{\frac{2}{3}}+c^{\frac{2}{3}}-a^{\frac{1}{3}}b^{\frac{1}{3}}-a^{\frac{1}{3}}c^{\frac{1}{3}}-b^{\frac{1}{3}}c^{\frac{1}{3}}$ by $a^{\frac{1}{3}}+b^{\frac{1}{3}}+c^{\frac{1}{3}}$.

9. $a^{\frac{1}{2}}-3a^{\frac{1}{3}}b^{\frac{1}{6}}+3a^{\frac{1}{6}}b^{\frac{1}{3}}-b^{\frac{1}{2}}$ by $a^{\frac{1}{3}}-2a^{\frac{1}{6}}b^{\frac{1}{6}}+b^{\frac{1}{3}}$.

10. $x^{\frac{5}{8}}+x^{\frac{1}{4}}y^{-\frac{3}{8}}+y^{-\frac{5}{8}}+x^{\frac{3}{8}}y^{-\frac{1}{4}}+x^{\frac{1}{8}}y^{-\frac{1}{2}}+x^{\frac{1}{2}}y^{-\frac{1}{8}}$ by $x^{\frac{3}{8}}+x^{\frac{1}{8}}y^{-\frac{1}{4}}-y^{-\frac{3}{8}}-x^{\frac{1}{4}}y^{-\frac{1}{8}}$.

Divide:

11. $a^{\frac{2}{3}}b^{-\frac{2}{3}}+1-a^{\frac{2}{3}}-b^{-\frac{2}{3}}$ by $1+b^{-\frac{1}{3}}+a^{\frac{1}{3}}+a^{\frac{1}{3}}b^{-\frac{1}{3}}$.

12. $x^{\frac{4}{3}}+a^{\frac{2}{3}}x^{\frac{2}{3}}+a^{\frac{4}{3}}$ by $x^{\frac{2}{3}}+a^{\frac{1}{3}}x^{\frac{1}{3}}+a^{\frac{2}{3}}$.

13. $x^2y^{-2}+x^{-2}y^2+2$ by $x^{\frac{2}{3}}y^{-\frac{2}{3}}+x^{-\frac{2}{3}}y^{\frac{2}{3}}-1$.

14. $1-x^{-\frac{1}{2}}+2x^{\frac{3}{4}}+x^{\frac{3}{2}}$ by $1+x^{-\frac{1}{4}}+x^{\frac{3}{4}}$.

15. $x+a$ by $x^{\frac{1}{5}}+a^{\frac{1}{5}}$.

16. $x^{-1}+y^{-1}$ by $x^{-\frac{1}{3}}+y^{-\frac{1}{3}}$.

17. $8a^{\frac{3}{2}}+b^{-\frac{3}{2}}-c+6a^{\frac{1}{2}}b^{-\frac{1}{2}}c^{\frac{1}{3}}$ by $2a^{\frac{1}{2}}+b^{-\frac{1}{2}}-c^{\frac{1}{3}}$.

Example 4. Find the square of $a^{\frac{1}{5}}-b^{\frac{1}{7}}+c^{\frac{1}{3}}$.

$$(a^{\frac{1}{5}}-b^{\frac{1}{7}}+c^{\frac{1}{3}})^2=(a^{\frac{1}{5}})^2+(-b^{\frac{1}{7}})^2+(c^{\frac{1}{3}})^2+2(a^{\frac{1}{5}})(-b^{\frac{1}{7}})+$$
$$2(a^{\frac{1}{5}})(c^{\frac{1}{3}})+2(-b^{\frac{1}{7}})(c^{\frac{1}{3}})=a^{\frac{2}{5}}+b^{\frac{2}{7}}+c^{\frac{2}{3}}-2a^{\frac{1}{5}}b^{\frac{1}{7}}+2a^{\frac{1}{5}}c^{\frac{1}{3}}-2b^{\frac{1}{7}}c^{\frac{1}{3}}$$

Find the square of:

18. $a^{\frac{1}{3}}+b^{\frac{1}{3}}$.

19. $a+a^{-1}$.

20. $a+a^{\frac{1}{2}}-a^{\frac{1}{3}}$.

21. $a^{\frac{1}{3}}-2a^{\frac{1}{2}}+a^{\frac{5}{6}}$.

22. $1+a^{-\frac{1}{3}}+a^{\frac{1}{3}}$.

23. $a^{\frac{3}{4}}-a^{\frac{1}{2}}b^{-\frac{1}{4}}+b^{\frac{1}{2}}$.

Example 5. Resolve into factors $a^{\frac{3}{2}}-b^{\frac{3}{2}}$:

$$a^{\frac{3}{2}}-b^{\frac{3}{2}}=(a^{\frac{1}{2}})^3-(b^{\frac{1}{2}})^3=(a^{\frac{1}{2}}-b^{\frac{1}{2}})\{(a^{\frac{1}{2}})^2+(a^{\frac{1}{2}})(b^{\frac{1}{2}})+(b^{\frac{1}{2}})^2\}$$
$$=(a^{\frac{1}{2}}-b^{\frac{1}{2}})(a+a^{\frac{1}{2}}b^{\frac{1}{2}}+b).$$

24. Do questions 3, 4, 7, 8, 16 and 17 by formulæ.

25. Multiply $a^{2n}+a^nb^{-n}+b^{-2n}$ by a^n-b^{-n}, using a formula.

26. Divide $a^{\frac{3n}{2}}-b^{-\frac{3n}{2}}$ by $a^{\frac{n}{2}}-b^{-\frac{n}{2}}$ by factors.

27. Find the L.C.M. of $a^{\frac{3}{2}}+b^{\frac{3}{2}}$, $a^{\frac{3}{2}}-b^{\frac{3}{2}}$, $a^{\frac{1}{2}}+b^{\frac{1}{2}}$, $a-b$ by factors.

28. Find the continued product of $1-x^{\frac{1}{2}}$, $1+x^{\frac{1}{2}}$, $1+x$.

Example 6. Simplify (i) $(x^{2^{n-1}})^2$, (ii) $(x^{3^{n-1}})^9$.

(i) $(x^{2^{n-1}})^2 = [x^{(2^{n-1})}]^2 = x^{2.2^{n-1}} = x^{2^{n-1+1}} = x^{2^n}$.

(ii) $(x^{3^{n-1}})^9 = [x^{(3^{n-1})}]^9 = x^{9.3^{n-1}} = x^{3^2.3^{n-1}}$

$= x^{3^{n-1+2}} = x^{3^{n+1}}$.

Simplify:

29. $(x^{2^{n+1}})^4$.

30. $(x^{3^{n+1}})^9$.

31. $27^n \div 3^{3n+1}$.

32. $[(a^m)^{m-\frac{1}{m}}]^{\frac{1}{m+1}}$.

33. $(x^{2^{n-1}} + a^{2^{n-1}})(x^{2^{n-1}} - a^{2^{n-1}})$.

34. $(a^{2^n} - b^{2^n}) \div (a^{2^{n-1}} + b^{2^{n-1}})$.

35. Compare $\left[(2^2)^2\right]^2$ and $2^{2^{2^2}}$.

36. Find the value of $3^m.3^n$ when $m = 2^4$ and $n = 2^3$.

Example 7. Simplify $\dfrac{2^{m+1}.3^{2m-n}.5^{m+n}.6^n}{6^m.10^{n+2}.15^m}$.

The expression $= \dfrac{2^{m+1}\,3^{2m-n}.5^{m+n}.(3^n.2^n)}{(3^m.2^m).(5^{n+2}.2^{n+2}).5^m.3^m}$

$= \dfrac{2^{m+n+1}.3^{2m+n-n}.5^{m+n}}{2^{m+n+2}.3^{m+m}.5^{m+n+2}}$

$= \left(\dfrac{2^{m+n+1}}{2^{m+n+2}}\right).\ \left(\dfrac{3^{2m}}{3^{2m}}\right).\ \left(\dfrac{5^{m+n}}{5^{m+n+2}}\right)$

$= \dfrac{1}{2}.1.\dfrac{1}{5^2} = \dfrac{1}{2} \cdot \dfrac{1}{25} = \dfrac{1}{50}$.

Simplify:

37. $\dfrac{2^n.6^{m+1}.10^{m-n}.15^{m+n-2}}{4^m.3^{2m+n}.25^{m-1}}$.

38. $\dfrac{3^{n+4} - 6.3^{n+1}}{3^{n+2}}$.

39. $\dfrac{2^n.(2^{n-1})^n}{2^{n+1}.2^{n-1}} \times \dfrac{2 \cdot 2^n}{(2^n)^n}$.

40. $\dfrac{3{\cdot}4 \times 10^4 \times 5{\cdot}7 \times 10^{-6} \times 4{\cdot}4 \times 10^{-3}}{1{\cdot}87 \times 10^{-4} \times 9{\cdot}5 \times 10^3 \times 3 \times 10^{-4}}$.

41. $\dfrac{6^{-n}\times 36^{2n-2}}{6^{3n-2}\times 12^{-1}}$.

42. $\dfrac{3^{n+1}}{(3^n)^{n+1}}\div\dfrac{9^{n+1}}{(3^{n+1})^{n-1}}$.

Prove that

***43.** $\dfrac{2^n+2^{n-1}}{2^{n+1}-2^n}=\dfrac{3}{2}$.

***44.** $\dfrac{5^3.2^{\frac{1}{4}}.10^{-\frac{1}{4}}}{15^{\frac{3}{4}}\cdot 6^{-\frac{3}{4}}.4^{\frac{3}{8}}}=25$.

Example 8. $\left(\dfrac{a-b}{a^{\frac{1}{2}}-b^{\frac{1}{2}}}-\dfrac{a^{\frac{3}{2}}-b^{\frac{3}{2}}}{a-b}\right)^{-1}$.

The expression $=\left[\dfrac{(a^{\frac{1}{2}}+b^{\frac{1}{2}})(a^{\frac{1}{2}}-b^{\frac{1}{2}})}{a^{\frac{1}{2}}-b^{\frac{1}{2}}}-\dfrac{(a^{\frac{1}{2}}-b^{\frac{1}{2}})(a+a^{\frac{1}{2}}b^{\frac{1}{2}}+b)}{(a^{\frac{1}{2}}-b^{\frac{1}{2}})(a^{\frac{1}{2}}+b^{\frac{1}{2}})}\right]^{-1}$

$=\left[(a^{\frac{1}{2}}+b^{\frac{1}{2}})-\dfrac{a+a^{\frac{1}{2}}b^{\frac{1}{2}}+b}{a^{\frac{1}{2}}+b^{\frac{1}{2}}}\right]^{-1}$

$=\left[\dfrac{(a^{\frac{1}{2}}+b^{\frac{1}{2}})^2-(a+a^{\frac{1}{2}}b^{\frac{1}{2}}+b)}{a^{\frac{1}{2}}+b^{\frac{1}{2}}}\right]^{-1}$

$=\left[\dfrac{a+2a^{\frac{1}{2}}b^{\frac{1}{2}}+b-a-a^{\frac{1}{2}}b^{\frac{1}{2}}-b}{a^{\frac{1}{2}}+b^{\frac{1}{2}}}\right]^{-1}$

$=\left[\dfrac{a^{\frac{1}{2}}b^{\frac{1}{2}}}{a^{\frac{1}{2}}+b^{\frac{1}{2}}}\right]^{-1}=\dfrac{a^{\frac{1}{2}}+b^{\frac{1}{2}}}{a^{\frac{1}{2}}b^{\frac{1}{2}}}$.

Simplify :

45. $\dfrac{\left(p+\dfrac{1}{q}\right)^m\left(p-\dfrac{1}{q}\right)^n}{\left(q+\dfrac{1}{p}\right)^m\left(q-\dfrac{1}{p}\right)^n}$.

46. $\dfrac{x-x^{-1}}{x-1}$.

47. $\dfrac{x^3-y^{-3}}{x-y^{-1}}$.

48. $\dfrac{x^2}{(1-x)^n}+\dfrac{2}{(1-x)^{n-1}}-\dfrac{1}{(1-x)^{n-2}}$.

49. $\left(\dfrac{x^a}{x^b}\right)^{a+b}\cdot\left(\dfrac{x^b}{x^c}\right)^{b+c}\cdot\left(\dfrac{x^c}{x^a}\right)^{c+a}$.

50. $(2x^{\frac{1}{2}}+x^{-\frac{1}{2}})^2-(2x^{\frac{1}{2}}-x^{-\frac{1}{2}})^2$.

*51. $\dfrac{x^2+y^2-x^{-2}-y^{-2}}{x^2y^2-x^{-2}y^{-2}}+\dfrac{(x-x^{-1})(y-y^{-1})}{xy+x^{-1}y^{-1}}$.

*52. $a^{-2}(bc^{-1}-b^{-1}c)+b^{-2}(ca^{-1}-c^{-1}a)+c^{-2}(ab^{-1}-a^{-1}b)$.

Prove that

*53. $\dfrac{y^{-1}}{x^{-1}+y^{-1}}+\dfrac{y^{-1}}{x^{-1}-y^{-1}}=\dfrac{2xy}{y^2-x^2}$.

*54. $\left(1-\dfrac{1}{n}\right)^{-n}=\left(1+\dfrac{1}{n-1}\right)^{n}$.

*55. Simplify $\dfrac{1}{1+a^{y-x}+a^{z-x}}+\dfrac{1}{1+a^{x-y}+a^{z-y}}+\dfrac{1}{1+a^{y-z}+a^{x-z}}$.

[*Hint.* $\dfrac{1}{1+a^{y-x}+a^{z-x}}=\dfrac{a^x}{a^x+a^y+a^z}$.]

*56. If $x=2^{\frac{1}{3}}-2^{-\frac{1}{3}}$, prove that $2x^3+6x=3$.

[*Hint.* Cube both sides $x=2^{\frac{1}{3}}-2^{-\frac{1}{3}}$, and put x for $(2^{\frac{1}{3}}-2^{-\frac{1}{3}})$.]

*57. If $a^l=b^m=c^n$ and $b^2=ac$, then $\dfrac{1}{l}+\dfrac{1}{n}=\dfrac{2}{m}$.

58. If $a^x=b^y$ and $b^x=a^y$, then $a=b$.

59. If $a^x=b$, $b^y=c$, $c^z=a$, then $xyz=1$.

*60. If $x^y=y^x$, prove that $\left(\dfrac{x}{y}\right)^{\frac{x}{y}}=x^{\frac{x}{y}-1}$

[*Hint.* Proceed analytically: $\left(\dfrac{x}{y}\right)^{\frac{x}{y}}=x^{\frac{x}{y}-1}$

if $\dfrac{(x^{\frac{x}{y}})}{(y^{\frac{x}{y}})}=\dfrac{(x^{\frac{x}{y}})}{x}$

if $y^{\frac{x}{y}}=x$

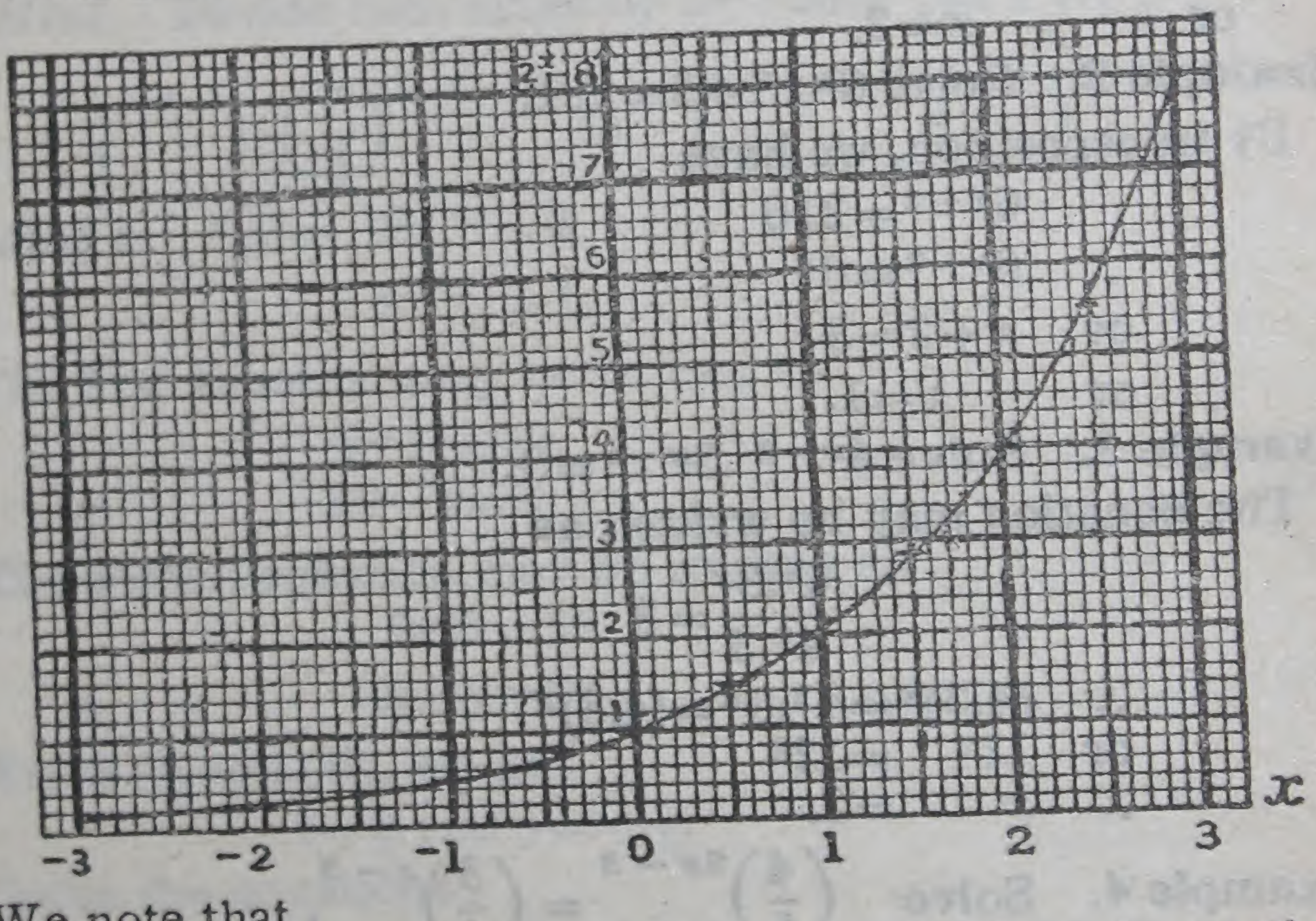
if $y^x=x^y$.]

8. To draw the graph of 2^x from $x=-3$ to $x=+3$.

Putting $x=-3,\ -2,\ -1,\ 0,\ \cdot5,\ 1,\ 1\cdot5,\ 2,\ 2\cdot5$ and 3 in 2^x and calculating its values, we prepare the following table :

x	-3	-2	-1	0	$\cdot5$	1	$1\cdot5$	2	$2\cdot5$	3
2^x	$\cdot125$	$\cdot25$	$\cdot5$	1	$1\cdot41$	2	$2\cdot83$	4	$5\cdot66$	8

Taking 1 cm. as a unit along x-axis and $\cdot5$ cm. as a unit along the other, and plotting these values, we obtain the following graph :



We note that

(i) 2^x increases as x increases. As there is no limit to the value of x, there is no limit to the value of 2^x.

(ii) For $x=0$, $2^x=1$,
for x positive, $2^x>1$
for x negative, $2^x<1$.

(iii) 2^x is positive for all values of x.

Ex. (i) From the graph, read off the approximate value of $2^{\cdot2}$ and $2^{-\cdot5}$

(ii) Use the graph to solve $2^x=5$.

9. Exponential Equations. The equations in which the unknown occurs as an index or exponent are called **exponential equations.** Thus $a^x = n$ is an exponential equation.

Example 1. Solve $9^x = 3^{x+2}$.

As $9^x = (3^2)^x = 3^{2x}$,

$\therefore$ the equation can be written as below:

$$3^{2x} = 3^{x+2}.$$

As the bases of this equation are equal, therefore its indices are also equal.

Hence $2x = x + 2$.

or $x = 2$.

Example 2. Solve $6^{x+2} + 5 = 221$.

By transposition, we have

$$6^{x+2} = 216$$

$\therefore$ $6^{x+2} = 6^3$

or $x + 2 = 3$

or $x = 1$.

Example 3. Solve $4^{x-2} \cdot 3^{x-1} = 3$.

The equation may be written as

$$\frac{4^x \cdot 3^x}{4^2 \cdot 3} = 3$$

$\therefore$ $(4.3)^x = 4^2 \cdot 3^2 = (4.3)^2$

or $12^x = 12^2$

$\therefore$ $x = 2$.

Example 4. Solve $\left(\frac{4}{5}\right)^{2x-3} = \left(\frac{5}{4}\right)^{x-3}$.

As $\frac{5}{4} = \frac{1}{\left(\frac{5}{4}\right)^{-1}} = \left(\frac{4}{5}\right)^{-1}$ $\therefore$ $\left(\frac{5}{4}\right)^{x-3} = \left(\frac{4}{5}\right)^{3-x}$

$\therefore$ $\left(\frac{4}{5}\right)^{2x-3} = \left(\frac{4}{5}\right)^{3-x}$

$\therefore$ $2x - 3 = 3 - x$

or $3x = 6$

or $x = 2$.

EXERCISE 73.

Solve the equations :

1. $2^{x+5}=4^{x+1}$. 2. $9^{x+1}=3^{x+7}$. 3. $5^{x+2}+6=631$.

4. $3^{2x-1}+5=86$. 5. $(\sqrt{6})^{x+4}=(\sqrt[3]{6})^{2x+3}$.

6. $(\sqrt[6]{16})^{2x+9}=(\sqrt[11]{64})^{2x+7}$. 7. $(\sqrt[4]{25})^{2x-1}=(\sqrt[6]{125})^{x+8}$.

8. $7^{x-1}.\ 3^{x-2}=7$. 9. $3^{x-2}.\ 5^{x-1}=5$. 10. $9^x=\frac{9}{3^x}$.

11. $5^{3x-4}=1$. [*Hint.* $1=5^0$.] 12. $a^{x-3}=b^{x-3}$.

[*Hint.* Divide both sides by b^{x-3} and put $1=\left(\frac{a}{b}\right)^0$.]

13. $\left(\frac{5}{7}\right)^{x-1}=\left(\frac{7}{5}\right)^{x-2}$. 14. $\left(\frac{3}{7}\right)^{2x-1}=\left(\frac{7}{3}\right)^{x-7}$.

Example 5. Solve $3^{2x-y}=9$

$5^{x+y}=1$.

The first equation may be written as

$$3^{2x-y}=3^2$$

or $$2x-y=2 \quad \ldots \quad \ldots \quad \text{(i)}$$

The second equation may be written as

$$5^{x+y}=5^0$$

$\therefore$ $$x+y=0 \quad \ldots \quad \ldots \quad \text{(ii)}$$

From (i) and (ii), we get

$x=\frac{2}{3}$ and $y=-\frac{2}{3}$.

Solve the equations :

*15. $\left.\begin{aligned} 3^x.\ 9^y &= 81 \\ 2x-y &= 8. \end{aligned}\right\}$

*16. $\left.\begin{aligned} 2^{x+1} &= 4^{y-3} \\ 3^{x-2} &= 9^{2y-3x}. \end{aligned}\right\}$

*17. $\left.\begin{aligned} a^{y-x} &= a^{z+1} \\ a^{2y+3} &= a^{3x+z} \\ x+y+z &= 3. \end{aligned}\right\}$

*18. $\left.\begin{aligned} a^{x+z}.\ a^{1-y} &= 1 \\ b^{y-z}.\ b^{2z-x} &= b \\ c^{y+1}.\ c^{x-z} &= c^3. \end{aligned}\right\}$

*19. $\left.\begin{aligned} 3^{x+y+z} &= 27^{x-y+z} \\ 4^{3y+2} &= 16^{x+z} \\ 2^{2x+2z+y} &= 4^{3x+y}. \end{aligned}\right\}$

*20. $\left.\begin{aligned} a^x.a^{y+1} &= a^7 \\ a^{2y}.\ a^{3x+5} &= a^{20}. \end{aligned}\right\}$

*21. $\left.\begin{aligned} 2^{x-y}.\ \ 3^{x-2y} &= 2^y \\ 3^{x-2}.\ \ 5^{x-1} &= 5^{3y+1}.\ 3^{3y}. \end{aligned}\right\}$

[*Hint.* Put the 1st equation in the form $(2.3)^x = (2.3)^{2y}$ and the second equation in the form $(3.5)^{x-2} = (3.5)^{3y}$.

*22. $\left.\begin{aligned} 3^{2x-y}.\ 4^{2x} &= 4^y. \\ 4^{3x+1}.\ 5^{3x-2y+1} &= 4^{2y}. \end{aligned}\right\}$

10. Theorem. If $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \ldots\ldots$

then each of the ratios $= \left(\frac{pa^n + qc^n + re^n + \ldots}{pb^n + qd^n + rf^n + \ldots}\right)^{\frac{1}{n}}$.

where $p, q, r \ldots\ldots\ldots n$ are any quantities whatever.

Let $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \ldots = k$.

Then $a = bk, \quad c = dk, \ e = fk\ldots\ldots$

$$pa^n = pb^n k^n$$

$$qc^n = qd^n k^n$$

$$re^n = rf^n k^n$$

etc., etc.

Hence $pa^n + qc^n + re^n + \ldots\ldots = (pb^n + qd^n + rf^n + \ldots)k^n$

$$\frac{pa^n + qc^n + re^n + \ldots}{pb^n + qd^n + rf^n + \ldots} = k^n.$$

Taking n^{th} root of both sides, we have

$$\left(\frac{pa^n + qc^n + re^n + \ldots}{pb^n + qd^n + rf^n + \ldots}\right)^{\frac{1}{n}} = (k^n)^{\frac{1}{n}}$$

$$= k = \frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \ldots$$

Example. If $\frac{a}{b} = \frac{c}{d} = \frac{e}{f}$, prove that each of these ratios is equal to $\sqrt{\left(\frac{3a^{-2} - 5c^{-2} + 7e^{-2}}{3b^{-2} - 5d^{-2} + 7f^{-2}}\right)^{-1}}$

Let $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = k$.

$\therefore \quad a = bk, \ c = dk, \ e = fk.$

$\therefore \quad 3a^{-2}=3b^{-2}k^{-2}$

$-5c^{-2}=-5d^{-2}k^{-2}$

$7e^{-2}=7f^{-2}k^{-2}.$

$\therefore \quad 3a^{-2}-5c^{-2}+7e^{-2}=k^{-2}(3b^{-2}-5d^{-2}+7f^{-2})$

$\therefore$ or $\left(\dfrac{3a^{-2}-5c^{-2}+7e^{-2}}{3b^{-2}-5d^{-2}+7f^{-2}}\right)=k^{-2}$

or $\left(\dfrac{3a^{-2}-5c^{-2}+7e^{-2}}{3b^{-2}-5d^{-2}+7f^{-2}}\right)^{-1}=(k^{-2})^{-1}=k^2.$

Taking the sq. root of both sides, we have

$$\sqrt{\left(\frac{3a^{-2}-5c^{-2}+7e^{-2}}{3b^{-2}-5d^{-2}+7f^{-2}}\right)^{-1}}=\sqrt{k^2}=k.$$

$$\therefore \quad \frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\sqrt{\left(\frac{3a^{-2}-5c^{-2}+7e^{-2}}{3b^{-2}-5d^{-2}+7f^{-2}}\right)^{-1}}.$$

EXERCISE 74.

If $\dfrac{a}{b}=\dfrac{c}{d}=\dfrac{e}{f}$, prove that each of these ratios is equal to

1. $\sqrt{\dfrac{a^2-c^2+e^2}{b^2-d^2+f^2}}$.

2. $\left(\dfrac{a^3+2c^3-3e^3}{b^3+2d^3-3f^3}\right)^{\frac{1}{3}}$.

3. $\sqrt{\dfrac{5a^2-7c^2+6e^2}{5b^2-7d^2+6f^2}}$.

4. $\left(\dfrac{3a^3-4c^3+5e^3}{3b^3-4d^3+5f^3}\right)^{\frac{1}{3}}$.

5. $\sqrt[4]{\left(\dfrac{2a^{-4}+5c^{-4}+7e^{-4}}{2b^{-4}+5d^{-4}+7f^{-4}}\right)^{-1}}$.

6. $\left(\dfrac{la^7-mc^7+ne^7}{lb^7-md^7+nf^7}\right)^{\frac{1}{7}}$.

If $\dfrac{a}{b}=\dfrac{c}{d}$, prove that

7. $\sqrt{\dfrac{a^2+c^2}{b^2+d^2}}=\dfrac{ma+nc}{mb+nd}$.

8. $\dfrac{(\sqrt{a}+\sqrt{b})^2}{(\sqrt{c}+\sqrt{d})^2}=\dfrac{a-b}{c-d}$.

9. $\dfrac{\sqrt{3a^2+4c^2}}{\sqrt[3]{5a^3-6c^3}}=\dfrac{\sqrt{3b^2+4d^2}}{\sqrt[3]{5b^3-6d^3}}$.

If $\dfrac{a}{b}=\dfrac{c}{d}=\dfrac{e}{f}$, prove that

10. $\sqrt{\dfrac{pa^2+qc^2+re^2}{pb^2+qd^2+rf^2}}=\sqrt[3]{\dfrac{ace}{bdf}}$.

11. $\sqrt[3]{\dfrac{a^2c-c^2e-e^2a}{b^2d-d^2f-f^2b}}=\sqrt{\dfrac{a^2-c^2-e^2}{b^2-d^2-f^2}}$.

CHAPTER XVII

INVOLUTION AND EVOLUTION

1. Involution is the process of finding the powers of quantities. The powers of monomials have already been treated. Here, we propose to discuss the method of finding the powers of binomials, trinomials and multinomials.

We already know that

$$(x+a)\,(x+b)=x^2+(a+b)\,x+ab \text{ and}$$

$$(x+a)\,(x+b)\,(x+c)=x^3+(a+b+c)x^2+(ab+bc+ca)x+abc.$$

Here we observe that

(i) the degree of the product is the same as the number of factors ;

(ii) the first term is the product of the first terms of the factors ;

(iii) the co-efficient of the second terms is the sum of the second terms of the factors ;

(iv) the co-efficient of the third terms is the sum of the products of the second terms of the factors, taken two at a time ;

(v) the last term is the product of the second terms of the factors.

Putting Σ for the sum of like terms, we have

$$(x+a)\,(x+b)\,(x+c)=x^3+x^2\Sigma a+x\Sigma ab+abc.$$

Similarly,

$$(x+a)(x+b)(x+c)(x+d)=x^4+x^3\Sigma a+x^2\Sigma ab+x\Sigma abc+abcd.$$

Example 1. Find the continued product of

$$(x+1)(x+2)(x+3)(x-4).$$

Putting a for 1, b for 2, c for 3 and d for -4, we get the product

$$=x^4+x^3\Sigma a+x^2\Sigma ab+x\Sigma abc+abcd.$$

$$\Sigma a=1+2+3-4=2,$$

$\Sigma ab = 2+3-4+6-8-12 = -13,$
$\Sigma abc = 6-8-24-12 = -38,$
$abcd = -24,$
$\therefore$ the product $= x^4+2x^3-13x^2-38x-24.$

Example 2. Find the continued product of
$(x+a+b)(x-a+b)(x+a-b).$

The expression $= \{x+(a+b)\}\{x+(-a+b)\}\{x+(a-b)\}.$
The co-efficient of x^2 in the product
$= (a+b)+(-a+b)+(a-b) = a+b,$
the co-efficient of x in the product
$= (a+b)(-a+b)+(a+b)(a-b)+(-a+b)(a-b)$
$= -a^2+b^2+a^2-b^2-a^2+2ab-b^2$
$= -(a-b)^2,$
the last term $= (a+b)(-a+b)(a-b)$
$= -(a^3-a^2b-ab^2+b^3).$
$\therefore$ the product
$= x^3+x^2(a+b)-x(a-b)^2-(a^3-a^2b-ab^2+b^3).$

EXERCISE 75.

Find the continued product of :

1. $(x-1)(x-2)(x+4).$
2. $(x-2)(x+3)(x+4).$
3. $(x+2)(x+3)(x+4).$
4. $(x-2)(x-3)(x-4).$
5. $(x-a)(x-b)(x-c).$
6. $(x+1)(x+2)(x-4)(x-5).$
7. $(x-2)(x-3)(x+4)(x+5).$
8. $(x-a)(x-b)(x-c)(x-d).$
9. $(x+p+q)(x+p-q)(x-p+q).$
10. $(x+y+1)(x+y-1)(x-y+1).$

Example 3. Expand $(x+a)^4.$

$(x+a)^4 = (x+a)(x+a)(x+a)(x+a)$
In this case the second term of each factor is a
The co-efficient of x^3 in the product $= a+a+a+a = 4a,$
the co-efficient of x^2 in the product
$= a.a+a.a+a.a+a.a+a.a+a.a = 6a^2,$
the co-efficient of x in the product
$= a.a.a+a.a.a+a.a.a+a.a.a = 4a^3,$
the last term $= a.a.a.a. = a^4.$

$\therefore (x+a)^4 = x^4+4x^3a+6x^2a^2+4xa^3+a^4.$

Expand as in example 3 :

11. $(x+a)^5$.
12. $(x+a)^6$.
13. $(x-a)^4$.
14. $(x-a)^5$.

2. Expansion of Binomials.

We already know that

$$\left.\begin{aligned}(x+a)^3 &= x^3+3x^2a+3xa^2+a^3\\ (x-a)^3 &= x^3-3x^2a+3xa^2-a^3\end{aligned}\right\}$$

$$\left.\begin{aligned}(x+a)^4 &= x^4+4x^3a+6x^2a^2+4xa^3+a^4\\ (x-a)^4 &= x^4-4x^3a+6x^2a^2-4xa^3+a^4\end{aligned}\right\}$$

$$\left.\begin{aligned}(x+a)^5 &= x^5+5x^4a+10x^3a^2+10x^2a^3+5xa^4+a^5\\ (x-a)^5 &= x^5-5x^4a+10x^3a^2-10x^2a^3+5xa^4-a^5.\end{aligned}\right\}$$

From the above cases, we observe that :

(i) The number of terms in the expansion is greater than the index of the binomial by one.

(ii) The first and the last terms in the expansion are respectively x and a, each raised to the same power as the binomial.

(iii) In each successive term, the index of x decreases and that of a increases by one.

(iv) The sum of the indices of x and a in any term is equal to the index of the binomial.

(v) The co-efficient of the first term is one, and that of each succeeding term is obtained by multiplying the co-efficient of the preceding term by the index of x in that term and dividing the product by the *number* of terms already obtained. The co-efficient of the last term is one.

(vi) The co-efficients of the terms, equi-distant from the middle, are equal.

(vii) If the sign between the two terms of the binomial is positive, all the terms in the expansion are positive, and if negative, then the odd terms are positive and the even terms are negative.

Example 4. Expand $(x+a)^8$.

The total number of terms in the expansion $=9$,

the first term $=x^8$,

the second term $=\frac{1.8}{1}x^7a=8x^7a$,

the third term $=\frac{8.7}{2}x^6a^2=28x^6a^2$,

the fourth term $=\frac{28.6}{3}x^5a^3=56x^5a^3$,

the fifth term $=\frac{56.5}{4}x^4a^4=70x^4a^4$.

The numerical co-efficients of the sixth, seventh, eighth and ninth terms are respectively 56, 28, 8 and 1.

Hence, we have

$$(x+a)^8=x^8+8x^7a+28x^6a^2+56x^5a^3+70x^4a^4+56x^3a^5+28x^2a^6+8xa^7+a^8.$$

Example 5. Expand $(2a-3b)^5$.

The total number of terms in the expansion $=6$,

the first term $=(2a)^5$,

the second term $=\frac{1.5}{1}(2a)^4(3b)=5(2a)^4(3b)$,

the third term $=\frac{5.4}{2}(2a)^3(3b)^2=10(2a)^3(3b)^2$.

The numerical co-efficients of the fourth, fifth and sixth terms are respectively 10, 5 and 1.

Hence, we have

$$(2a-3b)^5=(2a)^5-5(2a)^4(3b)+10(2a)^3(3b)^2-10(2a)^2(3b)+5(2a)(3b)^4-(3b)^5$$

$$=32a^5-240a^4b+720a^3b^2-1080a^2b^3+810ab^4-243b^5.$$

Expand :

15. $(x+1)^5$. **16.** $(a-b)^6$. **17.** $(p+q)^7$.

18. $(1-x)^6$. **19.** $(2+m)^5$. **20.** $(3-a)^4$.

21. $(2x-1)^5$. **22.** $(1-3x)^6$. **23.** $(1-x)^{10}$.

24. $(2+3x)^6$. **25.** $(3a-2b)^5$. **26.** $\left(a+\frac{1}{a}\right)^5$.

27. $(a+\frac{1}{2}b)^4$. **28.** $\left(x-\frac{1}{x}\right)^7$.

Simplify :

29. $(x+1)^4-(x-1)^4$. **30.** $(x-1)^5+(x+1)^5$.

31. $(x+a)^6-(x-a)^6$. **32.** $(3p+4q)^4-(3p-4q)^4$.

Example 6. Find the co-efficient of x^2 in the expansion of $(x+2)^3\ (x-1)^4$.

The given expression

$$=(x^3+6x^2+12x+8)(x^4-4x^3+6x^2-4x+1).$$

The co-efficient of x^2 can be obtained, by multiplying $(+6x^2)$ by $(+1)$, $(+12x)$ by $(-4x)$ and $(+8)$ by $(+6x^2)$.

Hence, the required co-efficient

$$=(+6)(+1)+(+12)(-4)+(+8)(+6)$$
$$=6-48+48=6.$$

Find the co-efficient of :

33. x^2 in the expansion of $(2x-5)^2(3x+1)^4$.

34. x^3 in the expansion of $(x+1)^5(x-2)^3$.

35. x^4 in the expansion of $(1-x)^6\left(2x+\frac{3}{x}\right)^2$.

***36.** x^5 in the expansion of $(1+3x+3x^2+x^3)^3$.

***37.** x^6 in the expansion of $(x^3-x^2-x+1)^3$.

NOTE. If x is small as compared with 1, $(1+x)^n$ is approximately equal to $1+nx+\frac{n(n-1)}{2}x^2$.

Example 7. Find an approximate value of $(1002)^5$.

We have $(1002)^5=(1000+2)^5=(1000)^5\left(1+\frac{2}{1000}\right)^5$.

$$\text{Now}\left(1+\frac{2}{1000}\right)^5 = 1+5\left(\frac{2}{1000}\right)+\frac{5.4}{2}\left(\frac{2}{1000}\right)^2+\ldots\ldots$$

$$=1+\frac{1}{100}+\frac{4}{100{,}000}+\ldots\ldots$$

$$=1{\cdot}01004\ldots\ldots\ldots\text{(approximately.)}$$

[In the expansion of $\left(1+\frac{2}{1000}\right)^5$, we neglect the terms beyond the third, because their value is very small as compared with 1.]

Hence $(1002)^5=(1000)^5\,(1{\cdot}01004)$

$$=10^{15}(1{\cdot}01004)$$

$$=1{,}010{,}040{,}000{,}000{,}000\ \text{(approximately.)}$$

Find an approximate value of :

***38.** $(1{\cdot}005)^5$. ***39.** $(1{\cdot}02)^{10}$. ***40.** $(1{\cdot}0003)^{25}$.

***41.** Re. 1 at 4% per annum compound interest after 10 years amounts to Rs. $(1+{\cdot}04)^{10}$. Calculate the value of this amount to 3 decimal places, by the above method.

EVOLUTION

3. Since $(+4)\times(+4)=+16$

and $(-4)\times(-4)=+16$

$\therefore\ \sqrt{+16}=+4$ as well as -4.

Similarly, $\sqrt{+x^2}=+x$ as well as $-x$.

Or briefly, $\sqrt{+16}=\pm4$ and $\sqrt{+x^2}=\pm x$.

Thus, *the square root of a positive quantity is both positive and negative.*

Since no two quantities, *both positive* or *both negative*, can give us a negative product, therefore a *negative quantity cannot have a real square root.*

Examples

(i) $\sqrt{+16a^4x^6}=\pm4a^2x^3$, for $(\pm4a^2x^3)^2=+16a^4x^6$.

(ii) $\sqrt[3]{-8a^3x^6}=-2ax^2$, for $(-2ax^2)^3=-8a^3x^6$.

(iii) $\sqrt[3]{+27x^6}=+3x^2$, for $(+3x^2)^3=+27x^6$.

(iv) $\sqrt[4]{+16a^8}=\pm2a^2$, for $(\pm2a^2)^4=+16a^8$.

(v) $\sqrt[5]{-32a^{10}} = -2a^2$, for $(-2a^2)^5 = -32a^{10}$.
(vi) $\sqrt[5]{+32a^{10}} = +2a^2$, for $(+2a^2)^5 = +32a^{10}$.

Hence the rules:

(i) Any *even* root of a positive quantity may be either positive or negative:

(ii) Any *odd* root of a quantity has the same sign as that of the quantity itself.

(iii) A negative quantity cannot have a real square root.

EXERCISE 76.

Find *mentally* the square root of:

1. $a^4b^{16}c^2$. 2. $64a^6b^8c^{10}$. 3. $\dfrac{81a^6b^2c^4}{x^{10}y^{12}z^{16}}$.

4. $1{\cdot}21\dfrac{a^{14}\,b^{18}}{c^6\,d^{10}}$. 5. $\dfrac{1{\cdot}44a^6(x-y)^4}{{\cdot}36\,b^8(x+y)^6}$.

Find *mentally* the cube root of:

6. $125a^6b^3c^9$. 7. $-64x^3b^6a^{15}$.

8. $343\cdot\dfrac{a^3b^6}{x^9y^{12}}$. 9. $-\dfrac{27}{64}\cdot\dfrac{a^6b^9}{x^{12}y^{18}}$.

Evaluate *mentally*:

10. $\sqrt[4]{16x^4y^8z^{12}}$. 11. $\sqrt[7]{-128a^{14}b^{21}}$.

12. $\sqrt[5]{\dfrac{32}{x^5y^{10}}}$. 13. $\sqrt[5]{-\dfrac{x^{10}y^{25}}{243}}$.

SQUARE ROOT

4. Since $a^2+2ab+b^2=(a+b)^2$
and $a^2-2ab+b^2=(a-b)^2$
$\therefore \sqrt{(a^2+2ab+b^2)}=\sqrt{(a+b)^2}=a+b$
and $\sqrt{(a^2-2ab+b^2)}=\sqrt{(a-b)^2}=a-b$.

Thus, we can write down the square root of a trinomial, when it can be put in the form of

$a^2+2ab+b^2$ or $a^2-2ab+b^2$.

Example 1. Find the square root of $25x^2-30xy+9y^2$.

Reducing the expression to the standard form, we have

$$\sqrt{(25x^2-30xy+9y^2)}=\sqrt{(5x)^2-2(5x)(3y)+(3y)^2}$$
$$=\sqrt{(5x-3y)^2}=\pm(5x-3y).$$

Example 2. Find the square root of $\frac{x^4}{4}-3x^2y+9y^2$.

Reducing the expression to the standard form, we have

$$\sqrt{\left(\frac{x^4}{4}-3x^2y+9y^2\right)}=\sqrt{\left(\frac{x^2}{2}\right)^2-2.\ \left(\frac{x^2}{2}\right)(3y)+(3y)^2}$$
$$=\sqrt{\left(\frac{x^2}{2}-3y\right)^2}=\pm\left(\frac{x^2}{2}-3y\right).$$

EXERCISE 77.

Find the square root of the following:

1. $4a^2-12ab+9b^2$.
2. $16x^4+40x^2+25$.
3. $9x^4+30x^2y^2+25y^4$.
4. $81a^2-18ab^2+b^4$.
5. a^6-2a^3+1.
6. $1-12x+36x^2$.
7. $x^2-x+\frac{1}{4}$.
8. $x^4y^2+x^2y+\frac{1}{4}$.
9. $m^4-\frac{2}{3}m^2+\frac{1}{9}$.
10. $\frac{a^2}{4b}-2+\frac{4b^2}{a^2}$.
11. $\frac{x^4}{y^4}-1+\frac{y^4}{4x^4}$.
12. $x^4-\frac{1}{2}x^2y^2+\frac{y^4}{16}$.
13. $\frac{1}{9}a^2-\frac{1}{6}ab+\frac{1}{16}b^2$.
14. $\frac{1}{4x^2}+\frac{1}{3y}+\frac{x^2}{9y^2}$.
15. $a^4-a^3b+\frac{1}{4}a^2b^2$.

Example 3. Find the square root of

$$4(a+b)^2-4(a^2-b^2)+(a-b)^2.$$

Reducing the expression to the standard form, we have

$$\{2(a+b)\}^2-2\{2(a+b)(a-b)\}+(a-b)^2.$$

$\therefore$ the square root $=2(a+b)-(a-b)$
$=a+3b.$

Find the square root of :

16. $4(x^2+1)^2+4(x^2+1)+1$.

17. $(3a^2+1)^2+14(3a^2+1)+49$.

18. $\left(\frac{x}{y}-2\right)^2+4\left(\frac{x}{y}-2\right)+4$.

19. $4(a+b)^2-12(a^2-b^2)+9(a-b)^2$.

20. $(a^4+1)^2-(a^4-1)^2$ 21. $\left(\frac{a+b}{a-b}\right)^2-2+\left(\frac{a-b}{a+b}\right)^2$.

22. $9(a+b)^2+6(a+b)(x+y)+x^2+2xy+y^2$.

Example 4. Find the square root of

$$x^2+\frac{1}{x^2}+12\left(x+\frac{1}{x}\right)+38.$$

The expression

$$=\left(x^2+2+\frac{1}{x^2}\right)+2.6\left(x+\frac{1}{x}\right)+36 \quad [\because 38=2+36.]$$

$$=\left(x+\frac{1}{x}\right)^2+2.6\left(x+\frac{1}{x}\right)+6^2=\left\{\left(x+\frac{1}{x}\right)+6\right\}^2$$

$\therefore$ the square root $=x+6+\frac{1}{x}$.

Example 5. Find the square root of

$$\left(x^2+\frac{1}{x^2}\right)^2-6\left(x+\frac{1}{x}\right)^2+21.$$

The expression $=\left(x^2+\frac{1}{x^2}\right)^2-6\left(x^2+\frac{1}{x^2}+2\right)+21$

$$=\left(x^2+\frac{1}{x^2}\right)^2-6\left(x^2+\frac{1}{x^2}\right)-12+21$$

$$=\left(x^2+\frac{1}{x^2}\right)^2-6\left(x^2+\frac{1}{x^2}\right)+9$$

$$=\left(x^2+\frac{1}{x^2}\right)^2-2.3\left(x^2+\frac{1}{x^2}\right)+3^2$$

$$=\left\{\left(x^2+\frac{1}{x^2}\right)-3\right\}^2$$

$\therefore$ the sq. root $=x^2-3+\frac{1}{x^2}$.

Find the square root of:

23. $x^2+\frac{1}{x^2}+8\left(x+\frac{1}{x}\right)+18.$

24. $x^2+\frac{1}{x^2}+4\left(x+\frac{1}{x}\right)+6.$

25. $x^4+\frac{1}{x^4}+2\left(x^2+\frac{1}{x^2}\right)+3.$

26. $x^2+\frac{1}{x^2}+2\left(x-\frac{1}{x}\right)-1.$

27. $\left(x+\frac{1}{x}\right)^2-4\left(x-\frac{1}{x}\right).$

28. $\left(x^2+\frac{1}{x^2}\right)^2-4\left(x+\frac{1}{x}\right)^2+12.$

29. $\left(x-\frac{1}{x}\right)^2+4\left(x+\frac{1}{x}\right)+8.$

30. $\left(x^2+\frac{1}{x^2}\right)^2-4\left(x-\frac{1}{x}\right)^2-4.$

Example 6. Find the square root of

$$\frac{y^2}{4}-xy+x^2-4x+4+2y.$$

Arranging the terms according to descending powers of x, we have

$$x^2-4x-xy+4+2y+\frac{y^2}{4}$$

$$=x^2-2x\left(2+\frac{y}{2}\right)+\left(2+\frac{y}{2}\right)^2=\left\{x-\left(2+\frac{y}{2}\right)\right\}^2$$

$\therefore$ the required square root $=x-2-\frac{y}{2}$.

Example 7. Find the square root of

$$x^4-6x^3+19x^2-30x+25.$$

Here, break up $19x^2$ into two parts, such that the first two terms and one of them may be a perfect square.

Thus, the expression

$$=x^4-6x^3+9x^2+10x^2-30x+25$$

$$=(x^2-3x)^2+2.5(x^2-3x)+5^2=\{(x^2-3x)+5\}^2$$

$\therefore$ the required square root $=x^2-3x+5$.

Find the square root of :

31. $x^4-4x^3+6x^2-4x+1$.

32. $4x^4+12x^3-11x^2-30x+25$.

33. $x^4+2x^3y+3x^2y^2+2xy^3+y^4$.

34. $9x^4-30x^3y+49x^2y^2-40xy^3+16y^4$.

35. $\frac{x^2}{y^2}+\frac{y^2}{x^2}+3+\frac{2x}{y}+\frac{2y}{x}$.

36. $x^2+\frac{y^2}{16}+z^2+\frac{xy}{2}-2xz-\frac{yz}{2}$.

37. $\frac{x^2}{y^2}+\frac{2x}{z}+\frac{y^2}{z^2}+\frac{2y}{x}+\frac{z^2}{x^2}+\frac{2z}{y}$.

38. $(ab+bc+ca)^2-4abc(a+c)$.

[*Hint.* $4abc(a+c)=4ac(ab+bc)$. Put x for $(ab+bc)$.]

39. $(2x^2+5xy+2y^2)(6x^2+5xy+y^2)(3x^2+7xy+2y^2)$.

40. $\left(\frac{y+z}{6}\right)^2+\frac{x(x-y-z)}{9}$.

[*Hint.* From the second fraction take $\frac{x^2}{9}$.]

41. $x^2+2x+1+\frac{2}{x-1}+\frac{1}{x^4-2x^2+1}$.

42. $(a-b)^4-2(a^2+b^2)(a-b)^2+2(a^4+b^4)$.

Find the fourth root of :

43. $x^4-4x^3+6x^2-4x+1$.

44. Shew that $(x+1)(x+2)(x+3)(x+4)+1$ is a perfect square.

45. Shew that $(x+3)(x+4)(x+5)(x+6)+1$ is a perfect square.

Example 8. Find the square root of $4^m+9^n-2^{m+1}.3^n$.

The expression $=(2^2)^m+(3^2)^n-2.2^m.3^n$

$=(2^m)^2+(3^n)^2-2.2^m.3^n$

$=(2^m-3^n)^2$

$\therefore$ the sq. root $=2^m-3^n$.

Find the square root of :

46. $x^{2m}-4x^{m+n}+4x^{2n}$.

47. $16^m+9^n+2^{2m+1}.3^n$.

18. $25x^{2m}-20x^{m+n}+4x^{2n}$.

5. General rule for finding the square root.

So far we have been extracting the square root of Algebraic expressions by factors, but there are expressions whose square roots cannot be conveniently determined by factors. Thus we stand in need of a method which is applicable in all cases.

Such a method is based upon the following formulæ :

$$(a+b)^2=a^2+(2a+b)b \quad \ldots \quad \ldots \quad \ldots \quad \text{(i)}$$

$$(a+b+c)^2=a^2+(2a+b)b+(2a+2b+c)c \quad \ldots \quad \text{(ii)}$$

$$(a+b+c+d)^2=a^2+(2a+b)b+(2a+2b+c)c+(2a+2b+2c+d)d \quad \ldots \quad \text{(iii)}$$

Suppose we have to find the square root of $a^2+2ab+b^2$.

Here, the first term of the sq. root is obviously $=\sqrt{a^2}=a$.

If a^2, or the square of the first term of the sq. root, is subtracted from the given expression, the remainder $=2ab+b^2$.

The second term of the square root (*i.e.*, b) can be derived from the second term of the expression by dividing it by twice the first term of the square root.

Now this term must be such as to give us $2ab+b^2$ when multiplied by $(2a+b)$, *i.e.*, twice the first term of the square root plus the second term.

As it is so, therefore b is the second term of the square root.

$\therefore$ $a+b$ is the square root of $a^2+2ab+b^2$.

The process explained above is abbreviated below :

Steps of process.

(i) The first term of the sq. root $= \sqrt{a^2} = a$.

(ii) The square of the first term of the sq. root $= a^2$.

(iii) When a^2 is subtracted from the given expression, the remainder $= +2ab + b^2$.

(iv) The second term of the sq. root $= +2ab \div 2a = +b$.

(v) $(+2a+b)b = +2ab+b^2$.

i.e., the 2nd term of the sq. root multiplied by the sum of twice the first term and the second term is equal to the remainder.

Abbreviated process.

$$\begin{array}{l|l} a & a^2+2ab+b^2 \big(a+b \\ & a^2 \\ \hline 2a+b & +2ab+b^2 \\ & +2ab+b^2 \\ \hline & \times \quad \times \end{array}$$

Example 1. Find the square root of $9x^2-24xy+16y^2$.

First term $= \sqrt{9x^2} = 3x$.

2nd term $= -24xy \div 6x = -4y$.

$$\begin{array}{l|l} 3x & 9x^2-24xy+16y^2 \big(3x-4y \\ & 9x^2 \\ \hline 6x-4y & -24xy+16y^2 \\ & -24xy+16y^2 \\ \hline & \times \quad \times \end{array}$$

If we have to find the square root of a multinomial, first we arrange it in descending powers of one of its letters and apply formula (i) or (ii), as the case may be. We find the first two terms of the square root, as explained in example 1. To find the third term, we divide the highest term of the second remainder by twice the first term of the square root already obtained ; the quotient thus got is the third term of the square root, provided it would satisfy the condition given below :

(twice 1st term + twice 2nd term + 3rd term) $\times$ 3rd term = the second remainder.

Example 2. Find the square root of $x^4-6x^3+4-12x+13x^2$.

The expression when arranged in descending powers of x is $x^4-6x^3+13x^2-12x+4$.

$$
\begin{array}{r|l}
x^4-6x^3+13x^2-12x+4 & x^2-3x+2 \\
x^4 & \\
\hline
-6x^3+13x^2-12x+4 & \\
-6x^3+9x^2 & \\
\hline
+4x^2-12x+4 & \\
+4x^2-12x+4 & \\
\hline
\times \quad \times \quad \times &
\end{array}
$$

(i) Here the 1st term $=\sqrt{x^4}=x^2$.

(ii) When $(x^2)^2$ or x^4 is subtracted from the expression, the first remainder $=-6x^3+13x^2-12x+4$.

(iii) $-6x^3 \div$ *twice* $x^2=-3x$, which is the 2nd term of the square root.

(iv) Multiply (*twice* x^2-3x) by $-3x$ and subtract the product from the first remainder; the result $+4x^2-12x+4$ is the second remainder.

(v) $+4x^2 \div$ *twice* $x^2=2$, which is the 3rd term of the square root.

(vi) Multiply [*twice* x^2+ *twice* $(-3x)+2$] by 2; the product is equal to the second remainder.

$\therefore$ x^2-3x+2 is the required sq. root.

Example 3. Find the square root of

$$x^4+\frac{4}{x^2}-2+4x-x^3+\frac{x^2}{4}.$$

In arranging the terms in descending powers of x, -2 must be placed before $\frac{4}{x^2}$.

Arranging the terms in descending powers of x and proceeding as usual, we have

$$x^4-x^3+\frac{x^2}{4}+4x-2+\frac{4}{x^2}\left(x^2-\frac{x}{2}+\frac{2}{x}\right.$$

x^2 | x^4

$2x^2-\frac{x}{2}$ | $-x^3+\frac{x^2}{4}+4x-2+\frac{4}{x^2}$

$2x^2-x+\frac{2}{x}$ | $-x^3+\frac{x^2}{4}+4x-2+\frac{4}{x^2}$

$\times \quad \times \quad \times$

$\therefore$ the square root $= x^2-\frac{x}{2}+\frac{2}{x}$.

Example 4. Find the square root of

$$x^{\frac{3}{2}}+xy^{-\frac{1}{2}}-2x^{\frac{5}{4}}y^{-\frac{1}{4}}-2x^{\frac{1}{2}}y^{\frac{1}{4}}+2x^{\frac{3}{4}}y^{\frac{1}{2}}+y$$

Arranging the terms according to descending powers of x and proceeding as usual, we have

$$x^{\frac{3}{4}}-x^{\frac{1}{2}}y^{-\frac{1}{4}}+y^{\frac{1}{2}}$$

$x^{\frac{3}{4}}$ | $x^{\frac{3}{2}}-2x^{\frac{5}{4}}y^{-\frac{1}{4}}+xy^{-\frac{1}{2}}+2x^{\frac{3}{4}}y^{\frac{1}{2}}-2x^{\frac{1}{2}}y^{\frac{1}{4}}+y$

$2x^{\frac{3}{4}}-x^{\frac{1}{2}}y^{-\frac{1}{4}}$ | $-2x^{\frac{5}{4}}y^{-\frac{1}{4}}+xy^{-\frac{1}{2}}$

$2x^{\frac{3}{4}}-2x^{\frac{1}{2}}y^{-\frac{1}{4}}+y^{\frac{1}{2}}$ | $+2x^{\frac{3}{4}}y^{\frac{1}{2}}-2x^{\frac{1}{2}}y^{\frac{1}{4}}+y$

$\times \quad \times \quad \times$

$\therefore$ the square root $= x^{\frac{3}{4}}-x^{\frac{1}{2}}y^{-\frac{1}{4}}+y^{\frac{1}{2}}$.

EXERCISE 78.

Find the square root of:

1. $16x^2-56xy+49y^2$.
2. $x^4+6x^3+15x^2+18x$
3. $16x^4-24x^3+25x^2-12x+4$.
4. $x^4-2ax^3+5a^2x^2-4a^3x+4a^4$
5. $x^6-12x^5+60x^4-160x^3+240x^2-192x+64$.

6. $a^2x^4+4ax^3+2(3a^2+2)x^2+12ax+9a^2$.

7. $\frac{x^4}{64}+\frac{x^3}{8}-x+1$.

8. $9x^4-18x^3+9x+\frac{9}{4}$.

9. $\frac{4}{9}x^4+x^3+\frac{59}{48}x^2+\frac{3}{4}x+\frac{1}{4}$.

10. $9x^2-24x+19-\frac{4}{x}+\frac{1}{4x^2}$.

11. $x^4-2x^3+\frac{3x^2}{2}-\frac{x}{2}+\frac{1}{16}$.

12. $\frac{25}{4}+16a^2b^2+9a^4b^4-5ab-6a^3b^3$.

13. $\frac{a^2}{4b^2}-\frac{a}{b}+\frac{4b^2}{a^2}-1+\frac{4b}{a}$.

14. $\frac{x^2}{y^2}+\frac{y^2}{x^2}-\frac{x}{y}+\frac{y}{x}-\frac{7}{4}$.

15. $\frac{9x^2}{4y^2}+\frac{5}{2}+\frac{y^2}{4x^2}-\frac{3x}{y}-\frac{y}{x}$.

16. $4x^4+9\left(x^2+\frac{1}{x^2}\right)+12x^2\left(x+\frac{1}{x}\right)+18$.

17. $a^6+\frac{1}{4a^6}+2-2\left(a^3+\frac{1}{2a^3}\right)$.

18. $x^{2m+2n}+x^{2m-2n}+\frac{1}{4}+2x^{2m}-x^{m+n}-x^{m-n}$.

19. $\frac{9}{4}x^{2m+2}+4x^{2m-2}+1-6x^{2m}+3x^{m+1}-4x^{m-1}$.

20. $\frac{x^4}{y^4}+\frac{y^4}{x^4}-4\left(\frac{x^2}{y^2}+\frac{y^2}{x^2}\right)+6$.

21. $(x+y)^4+x^4+y^4$.

22. $16x^{-\frac{3}{2}}+25+4x^{\frac{3}{2}}-12x^{\frac{3}{4}}-24x^{-\frac{3}{4}}$.

23. $\frac{1}{4}+\frac{1}{16}x^3+\frac{1}{3}x^{\frac{1}{2}}+\frac{1}{9}x-\frac{1}{6}x^2-\frac{1}{4}x^{\frac{3}{2}}$.

24. $x^2y^{-2}+2xy^{-1}+x^{-2}y^2+3+2x^{-1}y$.

25. $a^{\frac{4}{3}}+a^{-\frac{2}{3}}+1+2a^{\frac{1}{3}}+2a^{\frac{2}{3}}+2a^{-\frac{1}{3}}$.

Find the fourth root of:

26. $1-4x+6x^2-4x^3+x^4$.

27. $16x^4-96x^3y+216x^2y^2-216xy^3+81y^4$.

Example 5. Find the square root of $1.10^4+4.10^3+6.10^2+4.10+1$, taking 10 as the variable.

$1.10^2+2.10+1$

$1.10^4+4.10^3+6.10^2+4.10+1$ (

$1.10^2\times 1.10^2=$ 1.10^4

$+4.10^3+6.10^2$

$(2.10^2+2.10)\times 2.10=$ $+4.10^3+4.10^2$

$+2.10^2+4.10+1$

$(2.10^2+4.10+1)\times 1=$ $+2.10^2+4.10+1$

× × ×

The above process can be reduced to the arithmetical form by omitting the superfluous powers of 10, as shown below

$$\begin{array}{r} 1\quad 2\quad 1 \\ \hline 1\,4\,6\,4\,1 \\ 1\times 1= 1 \\ \hline 4\,6 \\ (20+2)\times 2= \quad 4\,4 \\ \hline 2\,4\,1 \\ (240+1)\times 1= \quad 2\,4\,1 \\ \hline *\,*\,* \end{array}$$

28. Find the square root of $4x^4+8x^3+8x^2+4x+1$, and *hence* extract the square root of 48841.

29. Find the square root of $x^4+2x^3+7x^2+6x+9$, and *hence* extract the square root of 12769.

Example 6. Find the square root of $1-4x$ up to 4 terms.

$$\begin{array}{l|l} & 1-4x \,(\; 1-2x-2x^2-4x^3 \\ 1 & 1 \\ \hline & -4x \\ 2-2x & -4x+4x^2 \\ \hline & -4x^2 \\ 2-4x-2x^2 & -4x^2+8x^34+x^4 \\ \hline & -8x^3-4x^4 \\ 2-4x-4x^2-4x^3 & -8x^3+16x^4+16x^5+16x^6. \end{array}$$

$\therefore$ the square root up to 4 terms $=1-2x-2x^2-4x^3$.

Find up to 4 terms the square root of :

30. a^2+x^2.

31. $1+x-x^2$.

Example 7. Find (i) what may be added to, (ii) what may be subtracted from, or (iii) what value may be given to m, so that the expression $16x^4-16x^3-20x^2+12x+m$ may be a perfect square ?

$$\begin{array}{l|l} & 16x^4-16x^3-20x^2+12x+m \,(\; 4x^2-2x-3 \\ 4x^2 & 16x^4 \\ \hline & -16x^3-20x^2 \\ 8x^2-2x & -16x^3+\;4x^2 \\ \hline & -24x^2+12x+m \\ 8x^2-4x-3 & -24x^2+12x+9 \\ \hline & m-9 \end{array}$$

Obviously, the given expression will be a perfect square

(i) if $-m+9$ is added to it, or

(ii) if $m-9$ is subtracted from it, or

(iii) if $m-9=0$, or $m=9$.

32. What may be added to, or subtracted from $9x^4+12x^3+7x^2+2x+a$, so that it may be a perfect square ?

33. For what value of x will $x^4-12x^3+38x^2-9x-5$ be a perfect square ?

34. Find the condition so that x^2+px+q may be a perfect square.

***35.** Find the values of a and b for which $x^4+4x^3+10x^2+ax+b$ is a perfect square.

CUBE ROOT

6. The cube root of an expression is found out by the application of the formulæ:

$$(a+b)^3=a^3+3a^2b+3ab^2+b^3 \quad \ldots \text{(i)}$$

$$(a-b)^3=a^3-3a^2b+3ab^2-b^3 \quad \ldots \text{(ii)}$$

Example. Find the cube root of $x^3-6x^2+12x-8$.

Reducing the expression to the standard form, we have

$x^3-6x^2+12x-8=(x)^3-3(2)(x)^2+3(2)^2(x)-(2)^3=(x-2)^3$.

$\therefore$ the cube root $=x-2$.

EXERCISE 79.

Find by formulæ the cube root of the following:

1. $x^3+12x^2+48x+64$.
2. $x^3-9x^2+27x-27$.
3. $8x^3-12x+\frac{6}{x}-\frac{1}{x^3}$.
4. $27x^3-27x^2y+9xy^2-y^3$.

CHAPTER XVIII

SURDS

1. When a root of a number *cannot be exactly obtained*, it is called an **irrational quantity**, or a **surd.** Thus, $\sqrt{2}$, $\sqrt[3]{4}$, $\sqrt{\frac{3}{5}}$ are all surds.

If a root of an Algebraical expression cannot be denoted without the use of a fractional index, it is called an **irrational quantity** or a **surd**.

Thus $\sqrt{x^2+xy+y^2}$, $\sqrt[3]{x^3+y^3}$, $\sqrt{\frac{x}{y}}$ are all surds.

In our mathematical work, we frequently meet with numbers and expressions which are in the *form of surds* but are *not really surds.*

Thus, $\sqrt{25}$ and $\sqrt{x^2+2xy+y^2}$ are in the surd form but they are not really surds; for $\sqrt{25}=5$ and $\sqrt{x^2+2xy+y^2}=x+y$.

A product partly rational and partly surd may be expressed as a *complete surd*, as illustrated below:

Example 1. $3\sqrt{5}=(3^2)^{\frac{1}{2}}.\ 5^{\frac{1}{2}}$

$=9^{\frac{1}{2}}\ .5^{\frac{1}{2}}=(9.5)^{\frac{1}{2}}$

$=\sqrt{9.5}=\sqrt{45}.$

Example 2. $4\sqrt[3]{6}=(4^3)^{\frac{1}{3}}\ .6^{\frac{1}{3}}$

$=64^{\frac{1}{3}}\ .6^{\frac{1}{3}}$

$=(64.6)^{\frac{1}{3}}$

$=\sqrt[3]{384.}$

A surd may sometimes be expressed as the product of a rational quantity and a surd, as illustrated below:

Example 3. $\sqrt{50}=\ \sqrt{25\times 2}$

$=(5^2\times 2)^{\frac{1}{2}}$

$=(5^2)^{\frac{1}{2}}\times 2^{\frac{1}{2}}$

$=5\sqrt{2.}$

Example 4. $\sqrt[3]{56}=\sqrt[3]{8\times7}=(2^3\times7)^{\frac{1}{3}}$

$=(2^3)^{\frac{1}{3}}\times7^{\frac{1}{3}}=2\times7^{\frac{1}{3}}$

$=2\sqrt[3]{7}.$

EXERCISE 80.

Express as a complete surd :

1. $4\sqrt{6}.$ 2. $5\sqrt[3]{4}.$ 3. $2\sqrt[4]{6}.$
4. $3\sqrt[4]{5}.$ 5. $2\sqrt[5]{3}.$ 6. $3\sqrt[5]{4}.$
7. $x\sqrt[n]{y}.$ 8. $x^2\sqrt[p]{y}.$ 9. $x^3\sqrt[5]{y^2}.$

Simplify :

10. $\sqrt{48}.$ 11. $\sqrt{63}.$ 12. $\sqrt{44}.$
13. $\sqrt[3]{250}.$ 14. $\sqrt[4]{567}.$ 15. $\sqrt[5]{80}.$
16. $\sqrt[3]{x^6y}.$ 17. $\sqrt[n]{x^{3n}.a}.$ 18. $\sqrt[a]{x^{4a}y^2}.$
19. $\sqrt[3]{-5120}.$ 20. $\sqrt[3]{250x^7y^4}.$

Surds are said to be **similar** when they have, or can be reduced so as to have the same irrational factors.

Thus, $3\sqrt{5}$, $7\sqrt{5}$ are similar surds, and $3\sqrt{125}$ and $7\sqrt{5}$ are also similar surds, for $3\sqrt{125}=15\sqrt{5}$.

The reduction of surds, to similar surds is necessary for the processes of addition and subtraction, as illustrated below

Example 1. $\sqrt[3]{128}+\sqrt[3]{54}=\sqrt[3]{64\times2}+\sqrt[3]{27\times2}$

$=\sqrt[3]{4^3\times2}+\sqrt[3]{3^3\times2}$

$=4\sqrt[3]{2}+3\sqrt[3]{2}$

$=\sqrt[3]{2(4+3)}=7\sqrt[3]{2}.$

Example 2. $\sqrt[4]{768}-\sqrt[4]{243}=\sqrt[4]{256\times3}-\sqrt[4]{81\times3}$

$=\sqrt[4]{4^4\times3}-\sqrt[4]{3^4\times3}$

$=4\sqrt[4]{3}-3\sqrt[4]{3}$

$=\sqrt[4]{3}(4-3)$

$=\sqrt[4]{3}.$

Surds are said to be of the **same order** when they have all got the same root-symbol. Thus, $\sqrt{7}$, $\sqrt{x-y}$, $\sqrt{5^3}$, $\sqrt{a^5}$ are all surds of the same (the *second*) order.

$\sqrt[3]{2}$, $\sqrt[3]{5}$, $\sqrt[3]{a^2}$, $\sqrt[3]{x+y}$ are all surds of the *third* order.

$\sqrt[n]{a}$, $\sqrt[n]{x^2}$, $\sqrt[n]{x^2-y^2}$ are all surds of the nth order.

Surds of the second order are called **quadratic surds** and of the third order are called **cubic surds.**

Example 3. Express $\sqrt{a^3}$, $\sqrt[3]{a^4}$ and $\sqrt[4]{a^5}$ as surds of the same order.

The given surds are of the 2nd, 3rd and 4th orders. The required order will be indicated by the L.C.M. of 2, 3, 4, or it will be of the 12th order. Thus,

$$\sqrt{a^3} = a^{\frac{3}{2}} = a^{\frac{18}{12}} = \sqrt[12]{a^{18}},$$

$$\sqrt[3]{a^4} = a^{\frac{4}{3}} = a^{\frac{16}{12}} = \sqrt[12]{a^{16}},$$

and $$\sqrt[4]{a^5} = a^{\frac{5}{4}} = a^{\frac{15}{12}} = \sqrt[12]{a^{15}}.$$

Hence the required surds are

$\sqrt[12]{a^{18}}$, $\sqrt[12]{a^{16}}$ and $\sqrt[12]{a^{15}}$.

Example 4. Which is greater $\sqrt[4]{5}$ or $\sqrt[3]{4}$?

The common order being 12th

$$\sqrt[4]{5} = 5^{\frac{1}{4}} = 5^{\frac{3}{12}} = \sqrt[12]{5^3} = \sqrt[12]{125}$$

and $$\sqrt[3]{4} = 4^{\frac{1}{3}} = 4^{\frac{4}{12}} = \sqrt[12]{4^4} = \sqrt[12]{256};$$

$\therefore$ $\sqrt[3]{4}$ is greater than $\sqrt[4]{5}$.

Example 5. Arrange $\sqrt{2}$, $\sqrt[3]{4}$ and $\sqrt[4]{6}$ in descending order of magnitude.

The common order being 12th

$$\sqrt{2} = 2^{\frac{1}{2}} = 2^{\frac{6}{12}} = \sqrt[12]{2^6} = \sqrt[12]{64},$$

$$\sqrt[3]{4} = 4^{\frac{1}{3}} = 4^{\frac{4}{12}} = \sqrt[12]{4^4} = \sqrt[12]{256},$$

$$\sqrt[4]{6} = 6^{\frac{1}{4}} = 6^{\frac{3}{12}} = \sqrt[12]{6^3} = \sqrt[12]{216}.$$

Hence the required arrangement is $\sqrt[3]{4}$, $\sqrt[4]{6}$ and $\sqrt{2}$.

EXERCISE 81.

Reduce to the simplest form:

1. $4\sqrt{5}+3\sqrt{20}-\sqrt{45}$.
2. $\sqrt{72}-2\sqrt{32}+\sqrt{8}$.
3. $\sqrt{12}-\sqrt{27}+\sqrt{75}$.
4. $2\sqrt[3]{4}+5\sqrt[3]{32}-\sqrt[3]{108}$.
5. $2\sqrt[3]{40}+3\sqrt[3]{625}-4\sqrt[3]{320}$.
6. $4\sqrt[3]{-27}+\sqrt[3]{-8}+5\sqrt[3]{-64}$.
7. $\sqrt{20x^3}+\sqrt{80x^3}-45\sqrt{x^2y}$.
8. $\sqrt{a^2x^3}-\sqrt{a^4x}-\sqrt{9a^4x}$.
9. $3\sqrt[3]{8a^4}-\sqrt[3]{125a^4}+\sqrt[3]{27a^4}$.

Reduce to surds of the same order:

10. $\sqrt{6}, \sqrt[4]{3}, \sqrt[3]{2}$.
11. $\sqrt[3]{2}, \sqrt[3]{4}, \sqrt[6]{8}$.
12. $\sqrt{x^3}, \sqrt[3]{x^2}, \sqrt[4]{x}$.

Which is greater:

13. $\sqrt[3]{5}$ or $\sqrt[4]{6}$?
14. $\sqrt{2}$ or $\sqrt[3]{3}$?
15. $\sqrt[3]{4}$ or $\sqrt[4]{5}$?
16. $\sqrt[3]{6}$ or $\sqrt[4]{10}$?

Arrange according to descending order of magnitude:

17. $\sqrt{2}, \sqrt[3]{4}, \sqrt[4]{5}$.
18. $\sqrt{3}, \sqrt[3]{5}, \sqrt[4]{6}$.
19. $2\sqrt{3}, 3\sqrt{2}, \sqrt[5]{7}$.
20. $\sqrt[3]{6}, \sqrt[4]{8}, \sqrt[6]{12}$.

2. Multiplication and division of surds.

Case I. When the surds are of the *same* order.

Example 1. $\sqrt[5]{8}\times\sqrt[5]{15}=8^{\frac{1}{5}}\times15^{\frac{1}{5}}=(8\times15)^{\frac{1}{5}}$

$$=(120)^{\frac{1}{5}}=\sqrt[5]{120}.$$

Example 2. $\sqrt[3]{10}\div\sqrt[3]{4}=10^{\frac{1}{3}}\div4^{\frac{1}{3}}$

$$=\left(\frac{10}{4}\right)^{\frac{1}{3}}=\sqrt[3]{\frac{5}{2}}.$$

Case II. When the surds are of *different* orders.

Example 3. $\sqrt[6]{8}\times\sqrt[4]{3}=8^{\frac{1}{6}}\times3^{\frac{1}{4}}=8^{\frac{2}{12}}\times3^{\frac{3}{12}}$

$$=(8^2)^{\frac{1}{12}}\times(3^3)^{\frac{1}{12}}=(8^2\times3^3)^{\frac{1}{12}}$$

$$=(64\times27)^{\frac{1}{12}}=\sqrt[12]{1728}.$$

Example 4. $\sqrt[9]{8} \div \sqrt[6]{6} = 8^{\frac{1}{9}} \div 6^{\frac{1}{6}}$

$$= 8^{\frac{2}{18}} \div 6^{\frac{3}{18}}$$

$$= (8^2)^{\frac{1}{18}} \div (6^3)^{\frac{1}{18}}$$

$$= (64)^{\frac{1}{18}} \div (216)^{\frac{1}{18}}$$

$$= \left(\frac{64}{216}\right)^{\frac{1}{18}} = \sqrt[18]{\frac{8}{27}}.$$

Case III. When the given surds have the *same* quantity under the radical sign.

Example 5. $\sqrt[4]{3} \times \sqrt[6]{3} = 3^{\frac{1}{4}} \times 3^{\frac{1}{6}}$

$$= 3^{\frac{1}{4} + \frac{1}{6}} = 3^{\frac{5}{12}}$$

$$= \sqrt[12]{3^5} = \sqrt[12]{243}.$$

Example 6. $\sqrt[3]{5} \div \sqrt[4]{5} = 5^{\frac{1}{3}} \div 5^{\frac{1}{4}}$

$$= 5^{\frac{1}{3} - \frac{1}{4}} = 5^{\frac{1}{12}}$$

$$= \sqrt[12]{5}.$$

Case IV. When simplification of surds is a necessary step before multiplication or division.

Example 7. $4.\sqrt{72} \times 2.\sqrt{45}$

$$= 4.\sqrt{6^2.2} \times 2.\sqrt{3^2.5}$$

$$= 4.6.\sqrt{2} \times 2.3\sqrt{5}$$

$$= 24\sqrt{2} \times 6\sqrt{5}$$

$$= 144\sqrt{2} \times \sqrt{5}$$

$$= 144\sqrt{10}.$$

Example 8. $6\sqrt{75} \div 2\sqrt{63}$

$$= 6\sqrt{5^2 \times 3} \div 2.\sqrt{3^2 \times 7}$$

$$= 6.5\sqrt{3} \div 2.3\sqrt{7}$$

$$= 30\sqrt{3} \div 6\sqrt{7}$$

$$= \frac{30\sqrt{3}}{6\sqrt{7}} = 5\sqrt{\frac{3}{7}}.$$

EXERCISE 82.

Simplify :—

1. $\sqrt[3]{7} \times \sqrt[3]{9}$.
2. $\sqrt[4]{5} \times \sqrt[4]{12}$.
3. $\sqrt[5]{8} \times \sqrt[5]{18}$.
4. $\sqrt[5]{12} \div \sqrt[5]{8}$.
5. $\sqrt[6]{16} \div \sqrt[6]{12}$.
6. $\sqrt[4]{20} \div \sqrt[4]{15}$.
7. $\sqrt[3]{6} \times \sqrt[4]{9}$.
8. $\sqrt{15} \times \sqrt[3]{12}$.
9. $\sqrt[3]{21} \times \sqrt[5]{6}$.
10. $\sqrt[3]{20} \div \sqrt[4]{15}$.
11. $\sqrt[5]{18} \div \sqrt[6]{24}$.
12. $\sqrt[4]{48} \div \sqrt[8]{72}$.
13. $\sqrt{36} \times \sqrt{54}$.
14. $\sqrt{96} \times \sqrt{88}$.
15. $\sqrt[3]{128} \times \sqrt[3]{54}$.
16. $5\sqrt[3]{49} \times \sqrt[3]{14}$.
17. $\sqrt{18} \div \sqrt{200}$.
18. $5\sqrt[3]{6} \div 3\sqrt{10}$.
19. $\sqrt{3}.\ \sqrt[3]{2}.\ \sqrt[6]{4}$.
20. $\sqrt{8}.\ \sqrt[4]{3}.\ \sqrt[3]{4}$.
21. $\sqrt{2}.\ \sqrt[3]{3}.\ \sqrt[6]{5}$.
22. $5\sqrt{12} \div 10\sqrt{27}$.
23. $\sqrt[3]{56} \div \sqrt[3]{48}$.
24. $x\sqrt[3]{x^4y^5}$.
25. $2\sqrt{24} \times 3\sqrt[4]{18} \times 4\sqrt[6]{24}$.

3. **Compound Surds.** An expression consisting of two or more simple surds connected by the signs + or − is called a **compound surd.** Thus. $6\sqrt[3]{7}$ and $2\sqrt[5]{8}$ being simple surds, $6\sqrt[3]{7}+2\sqrt[5]{8}$ and $6\sqrt[3]{7}-2\sqrt[5]{8}$ are compound surds.

The process of multiplication of compound surds is similar to that of compound algebraic expressions.

Example 1. Multiply $4\sqrt{3}+3\sqrt{2}$ by $2\sqrt{5}-5\sqrt{3}$.

$(4\sqrt{3}+3\sqrt{2})(2\sqrt{5}-5\sqrt{3})$.

$=4\sqrt{3} \times 2\sqrt{5}-4\sqrt{3} \times 5\sqrt{3}+3\sqrt{2} \times 2\sqrt{5}-3\sqrt{2} \times 5\sqrt{3}$.

$=8\sqrt{15}-60+6\sqrt{10}-15\sqrt{6}$.

Example 2. Find the square of $(5\sqrt{3}-6\sqrt{2})$.

$(5\sqrt{3}-6\sqrt{2})^2=(5\sqrt{3})^2-2.5\sqrt{3}.6\sqrt{2}+(6\sqrt{2})^2$

$=75-60\sqrt{6}+72$

$=147-60\sqrt{6}$.

EXERCISE 83.

Multiply :—

1. $\sqrt{2}+\sqrt{3}$ by $\sqrt{5}+\sqrt{7}$.
2. $\sqrt{5}+\sqrt{6}$ by $\sqrt{3}-\sqrt{2}$.
3. $4\sqrt{2}-5\sqrt{3}$ by $2\sqrt{2}+3\sqrt{3}$.
4. $3\sqrt{5}+6\sqrt{3}$ by $4\sqrt{7}-2\sqrt{5}$.
5. $6-2\sqrt{6}$ by $5-2\sqrt{6}$.
6. $5\sqrt{2}-2\sqrt{3}$ by $5\sqrt{2}-2\sqrt{3}$.

7. $6\sqrt{3}+3\sqrt{5}$ by $6\sqrt{3}-3\sqrt{5}$.

8. $\sqrt{5}+\sqrt{3}+\sqrt{6}$ by $\sqrt{5}+\sqrt{3}-\sqrt{6}$.

9. $\sqrt[3]{4}+\sqrt[3]{9}+\sqrt[3]{36}$ by $\sqrt[3]{2}+\sqrt[3]{3}$.

Find the continued product of :

10. $(2-\sqrt{3})$, $(\sqrt{3}+2)$, $(3-\sqrt{2})$.

11. $(4+\sqrt{6})$, $(\sqrt{3}+1)$, $(\sqrt{6}-\sqrt{3})$.

12. $(1+\sqrt{2}+\sqrt{3})$, $(1+\sqrt{2}-\sqrt{3})$, $(1-\sqrt{2}+\sqrt{3})$, $(-1+\sqrt{2}+\sqrt{3})$.

Find the square of :

13. $2\sqrt{6}+3\sqrt{5}$.

14. $3\sqrt{7}-4\sqrt{6}$.

15. $3\sqrt{3}-4$.

16. $x-\sqrt{x^2-1}$.

17. $a^2+\sqrt{a^2-b^2}$.

18. $\sqrt{a^2+2b^2}-\sqrt{a^2-2b^2}$.

19. $2\sqrt{8x-1}+3\sqrt{1-4x}$.

20. $3\sqrt{a^2+b^2}+4\sqrt{a^2-b^2}$.

21. Find the cube of $1+2\sqrt{x}$.

4. Rationalising Factor. When the product of two irrational expressions or surds is rational, each is said to be the **rationalising factor** of the other.

Thus $\sqrt[3]{5}$ is the rationalising factor of $\sqrt[3]{25}$ and $\sqrt[3]{25}$ is the rationalising factor of $\sqrt[3]{5}$ for $\sqrt[3]{25}\times\sqrt[3]{5}=\sqrt[3]{125}=5$, and $(\sqrt{5}+\sqrt{3})$ is the rationalising factor of $(\sqrt{5}-\sqrt{3})$ and $(\sqrt{5}-\sqrt{3})$ is the rationalising factor of $(\sqrt{5}+\sqrt{3})$, for $(\sqrt{5}+\sqrt{3})(\sqrt{5}-\sqrt{3})=5-3=2$.

The rationalising factor of a monomial surd can be found easily, as illustrated below :

The rationalising factor of $\sqrt[3]{a^2}$ or $a^{\frac{2}{3}}$ is obviously $a^{\frac{1}{3}}$ or $\sqrt[3]{a}$, for $\sqrt[3]{a^2}\times\sqrt[3]{a}=a^{\frac{2}{3}}\times a^{\frac{1}{3}}=a^{\frac{2}{3}+\frac{1}{3}}=a$.

Similarly, the rationalising factor of $\sqrt[3]{a^2}.\ \sqrt{b}.\ \sqrt[3]{c}$ or $a^{\frac{2}{3}}b^{\frac{1}{2}}c^{\frac{1}{3}}$ is $a^{\frac{1}{3}}b^{\frac{1}{2}}c^{\frac{2}{3}}$ or $\sqrt[3]{a}.\ \sqrt{b}.\ \sqrt[3]{c^2}$, for $\sqrt[3]{a^2}.\ \sqrt{b}.\ \sqrt[3]{c}\times\sqrt[3]{a}.\ \sqrt{b}.\ \sqrt[3]{c^2}$

$$=a^{\frac{2}{3}}b^{\frac{1}{2}}c^{\frac{1}{3}}\times a^{\frac{1}{3}}b^{\frac{1}{2}}c^{\frac{2}{3}}$$
$$=a^{\frac{2}{3}+\frac{1}{3}}\ b^{\frac{1}{2}+\frac{1}{2}}\ c^{\frac{1}{3}+\frac{2}{3}}$$
$$=a.b.c.$$

From these and similar examples, we can establish the following rule for finding the rationalising factor of monomial surds:

Rule. (i) **Change the radical sign of the surd into the fractional index.**

(ii) Find the *least fraction* which when added to this index will make it an integer—such a fraction is the index of the rationalising factor.

Example 1. Find the rationalising factor of $7\sqrt[5]{2^3}.\sqrt[4]{3^5}.\sqrt[3]{4^2}$.

$7\sqrt[5]{2^3}.\ \sqrt[4]{3^5}.\ \sqrt[3]{4^2} = 7.2^{\frac{3}{5}}.\ 3^{\frac{5}{4}}.\ 4^{\frac{2}{3}}$.

Since $\frac{3}{5}$ is the index of 2 and $\frac{3}{5}+\frac{2}{5}=1$,

$\therefore\ 2^{\frac{2}{5}}$ or $\sqrt[5]{2^2}$ is the rationalising factor of $\sqrt[5]{2^3}$.

Since $\frac{5}{4}$ is the index of 3 and $\frac{5}{4}+\frac{3}{4}=2$,

$\therefore\ 3^{\frac{3}{4}}$ or $\sqrt[4]{3^3}$ is the rationalising factor of $\sqrt[4]{3^5}$.

Since $\frac{2}{3}$ is the index of 4 and $\frac{2}{3}+\frac{1}{3}=1$,

$\therefore\ 4^{\frac{1}{3}}$ or $\sqrt[3]{4}$ is the rationalising factor of $\sqrt[3]{4^2}$.

Or the rationalising factor of $7.\sqrt[5]{2^3}.\sqrt[4]{3^5}.\sqrt[3]{4^2}$ is $\sqrt[5]{2^2}.\sqrt[4]{3^3}.\sqrt[3]{4}$.

The rationalising factor of a binomial quadratic surd can be found by merely changing the sign between the two terms, from $-$ into $+$ and from $+$ into $-$.

Example 2. Find the rationalising factor of $a\sqrt{x}+b\sqrt{y}$.

Since $(a\sqrt{x}+b\sqrt{y})(a\sqrt{x}-b\sqrt{y})=(a\sqrt{x})^2-(b\sqrt{y})^2$

$=a^2x-b^2y$,

$\therefore\ (a\sqrt{x}-b\sqrt{y})$ is the rationalising factor of $(a\sqrt{x}+b\sqrt{y})$.

The rationalising factor of a binomial qua dratic surd is called its **conjugate.**

EXERCISE 84.

Write down *mentally* the rationalising factor of:

1. $2\sqrt{2}$, $\sqrt{45}$, $\sqrt{54}$, $\sqrt{a^3b}$, $\sqrt[3]{a}$, $\sqrt{x-1}$, $\sqrt[4]{a^2b^3}$

2. $\sqrt{3}.\sqrt{2}$. 3. $\sqrt[3]{4}.\sqrt{3}$. 4. $\sqrt{2}.\sqrt{3}.\sqrt[3]{6}$.

5. $\sqrt{3}+\sqrt{2}$. **6.** $\sqrt{5}-\sqrt{3}$ **7.** $\sqrt{c}-\sqrt{b}$.

8. $a+\sqrt{b}$. **9.** $\sqrt{5}.\sqrt[3]{6}.\sqrt[4]{7^3}$. **10.** $\sqrt[3]{5}.\sqrt[3]{6^2}.\sqrt[4]{7^3}$.

11. $a\sqrt{x}-b\sqrt{y}$. **12.** $4\sqrt{3}+2\sqrt{5}$. **13.** $3\sqrt{2}-2\sqrt{3}$.

14. $\sqrt{a+x}-\sqrt{a-x}$. **15.** $x+\sqrt{x^2-1}$.

16. $\sqrt{x^2+1}-\sqrt{x^2-1}$.

Example 3. Simplify $5\sqrt{3}\div 4\sqrt{2}$.

$$5\sqrt{3}\div 4\sqrt{2}=\frac{5\sqrt{3}}{4\sqrt{2}}.$$

Rationalising the denominator, we get

$$\frac{5\sqrt{3}}{4\sqrt{2}}=\frac{5\sqrt{3}\times\sqrt{2}}{4\sqrt{2}\times\sqrt{2}}=\frac{5\sqrt{6}}{4.2}=\frac{5\sqrt{6}}{8}.$$

Example 4. Given $\sqrt{3}=1{\cdot}732$, find the value of $\frac{8}{2\sqrt{3}}$ up to 3 places of decimals.

Rationalising the denominator of this fraction, we have

$$\frac{8}{2\sqrt{3}}=\frac{8\times\sqrt{3}}{2\sqrt{3}\times\sqrt{3}}=\frac{8\sqrt{3}}{2.3}=\frac{8\sqrt{3}}{6}$$

$$=\frac{8\times 1{\cdot}732}{6}=\frac{13{\cdot}856}{6}=2{\cdot}309.$$

Simplify :

17. $5\sqrt{3}\div 7\sqrt{5}$. **18.** $7\sqrt[3]{4}\div 5\sqrt[3]{2}$.

19. $4\sqrt[3]{5}\div 4\sqrt[3]{2}$. **20.** $2\sqrt[4]{5}\div 6\sqrt[4]{3}$,

Given $\sqrt{2}=1{\cdot}414$, $\sqrt{3}=1{\cdot}732$ and $\sqrt{5}=2{\cdot}236$ find the value of the following fractions up to 3 places of decimals:

21. $\frac{4}{5\sqrt{3}}$. **22.** $\frac{3}{2\sqrt{2}}$. **23.** $\frac{7}{4\sqrt{3}}$.

24. $\frac{1}{3\sqrt{5}}$. **25.** $\frac{3}{2\sqrt{5}}$. **26.** $\frac{6}{5\sqrt{3}}$.

Example 5. Given $\sqrt{3}=1{\cdot}732$, find the value of $\frac{2+\sqrt{3}}{2-\sqrt{3}}$.

Rationalising the denominator of $\frac{2+\sqrt{3}}{2-\sqrt{3}}$, we have

$$\text{the fraction} = \frac{(2+\sqrt{3})(2+\sqrt{3})}{(2-\sqrt{3})(2+\sqrt{3})}$$
$$= \frac{4+3+4\sqrt{3}}{4-3} = 7+4\sqrt{3}$$
$$= 7+4\times 1{\cdot}732 = 7+6{\cdot}928$$
$$= 13{\cdot}928.$$

Example 6. Rationalise the denominator of $\dfrac{a+\sqrt{a^2-1}}{a-\sqrt{a^2-1}}$.

$$\text{The fraction} = \frac{(a+\sqrt{a^2-1})(a+\sqrt{a^2-1})}{(a-\sqrt{a^2-1})(a+\sqrt{a^2-1})}$$
$$= \frac{a^2+a^2-1+2a\sqrt{a^2-1}}{a^2-(a^2-1)}$$
$$= \frac{2a^2-1+2a\sqrt{a^2-1}}{a^2-a^2+1} = 2a^2+2a\sqrt{a^2-1}-1.$$

Given $\sqrt{2}=1{\cdot}414$, $\sqrt{3}=1{\cdot}732$, $\sqrt{5}=2{\cdot}236$ and $\sqrt{6}=2{\cdot}449$, find the value of :

27. $\dfrac{\sqrt{2}+1}{\sqrt{2}-1}$.
28. $\dfrac{1}{\sqrt{3}-2}$.
29. $\dfrac{\sqrt{5}+2}{\sqrt{5}-2}$.
30. $\dfrac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}$.
$\dfrac{7}{3\sqrt{3}-2\sqrt{2}}$.
32. $\dfrac{15}{4\sqrt{3}-3\sqrt{2}}$.
33. Solve $x\sqrt{2}+x=7$ up to 3 places of decimals.
34. Evaluate $\left(x+\dfrac{1}{x}\right)^2$ if $x=\sqrt{3}-\sqrt{2}$.
35. Evaluate $x^4+\dfrac{1}{x^4}$ when $x=3+\sqrt{8}$.

Reduce to an equivalent fraction with a rational denominator :

36. $\dfrac{2+3\sqrt{2}}{3-2\sqrt{2}}$.
37. $\dfrac{4\sqrt{2}+\sqrt{5}}{3\sqrt{3}-2\sqrt{2}}$.
38. $\dfrac{a+\sqrt{a^2-b^2}}{a-\sqrt{a^2-b^2}}$.
39. $\dfrac{\sqrt{x+1}+\sqrt{x-1}}{\sqrt{x+1}-\sqrt{x-1}}$.
40. $\dfrac{\sqrt{x^2+1}+\sqrt{x^2-1}}{\sqrt{x^2+1}-\sqrt{x^2-1}}$.

The rationalising factor of a binomial cubic surd can be found by the application of the identity $(a\pm b)(a^2\mp ab+b^2)$ $=a^3\pm b^3$.

Example 7. Find the rationalising factor of $\sqrt[3]{a}+\sqrt[3]{b}$.

$\sqrt[3]{a}+\sqrt[3]{b}=a^{\frac{1}{3}}+b^{\frac{1}{3}}$.

Since $\left(a^{\frac{1}{3}}+b^{\frac{1}{3}}\right)\left\{\left(a^{\frac{1}{3}}\right)^2-a^{\frac{1}{3}}b^{\frac{1}{3}}+\left(b^{\frac{1}{3}}\right)^2\right\}$

$$=\left(a^{\frac{1}{3}}\right)^3+\left(b^{\frac{1}{3}}\right)^3=a+b,$$

$\therefore \left(a^{\frac{1}{3}}\right)^2-a^{\frac{1}{3}}b^{\frac{1}{3}}+\left(b^{\frac{1}{3}}\right)^2$ or $\sqrt[3]{a^2}-\sqrt[3]{ab}+\sqrt[3]{b^2}$ is the required rationalising factor.

Write down *mentally* the rationalising factor of :

41. $\sqrt[3]{5}-\sqrt[3]{2}$. **42.** $\sqrt[3]{4}+\sqrt[3]{3}$. **43.** $\sqrt[3]{6}-\sqrt[3]{5}$.

44. $\sqrt[3]{6}+\sqrt[3]{3}$. **45.** $\sqrt[3]{x}-\sqrt[3]{y}$. **46.** $\sqrt[3]{x}+\sqrt[3]{y}$.

The rationalising factors of homogeneous binomial surds of higher orders can be found by the application of the following identities :

(i) $(a-b)(a^{n-1}+a^{n-2}b+a^{n-3}b^2+\ldots+ab^{n-2}+b^{n-1})=a^n-b^n$, for all values of n.

(ii) $(a+b)(a^{n-1}-a^{n-2}b+a^{n-3}b^2-\ldots+ab^{n-2}-b^{n-1})=a^n-b^n$ when n is *even*.

(iii) $(a+b)(a^{n-1}-a^{n-2}b+a^{n-3}b^2-\ldots-ab^{n-2}+b^{n-1})=a^n+b^n$ when n is *odd*.

Example 8. Find the rationalising factor $\sqrt[5]{a}-\sqrt[5]{b}$.

$$\sqrt[5]{a}-\sqrt[5]{b}=a^{\frac{1}{5}}-b^{\frac{1}{5}}.$$

In $(x-y)(x^4+x^3y+x^2y^2+xy^3+y^4)\equiv x^5-y^5$,

if we put $a^{\frac{1}{5}}$ for x and $b^{\frac{1}{5}}$ for y, we get

$$(a^{\frac{1}{5}}-b^{\frac{1}{5}})(a^{\frac{4}{5}}+a^{\frac{3}{5}}b^{\frac{1}{5}}+a^{\frac{2}{5}}b^{\frac{2}{5}}+a^{\frac{1}{5}}b^{\frac{3}{5}}+b^{\frac{4}{5}})$$

$$=(a^{\frac{1}{5}})^5-(b^{\frac{1}{5}})^5=a-b,$$

$\therefore \quad (a^{\frac{4}{5}}+a^{\frac{3}{5}}b^{\frac{1}{5}}+a^{\frac{2}{5}}b^{\frac{2}{5}}+a^{\frac{1}{5}}b^{\frac{3}{5}}+b^{\frac{4}{5}})$, or

$\sqrt[5]{a^4}+\sqrt[5]{a^3b}+\sqrt[5]{a^2b^2}+\sqrt[5]{ab^3}+\sqrt[5]{b^4}$ is the required rationalising factor.

Write down *mentally* the rationalising factor of:

47. $\sqrt[4]{3}-\sqrt[4]{2}$.
48. $\sqrt[5]{3}-\sqrt[5]{2}$.
49. $\sqrt[6]{5}-\sqrt[6]{3}$.
50. $\sqrt[4]{3}+\sqrt[4]{2}$.
51. $\sqrt[5]{3}+\sqrt[5]{2}$.
52. $\sqrt[6]{5}+\sqrt[6]{3}$.

Example 9. Find the rationalising factor of $1+\sqrt{2}+\sqrt{3}$.

$$1+\sqrt{2}+\sqrt{3}=\{(1+\sqrt{2})+\sqrt{3}\}.$$

Multiplying the expression by $(1+\sqrt{2})-\sqrt{3}$, we get

$$\{(1+\sqrt{2})+\sqrt{3}\}\ \{(1+\sqrt{2})-\sqrt{3}\}=(1+\sqrt{2})^2-(\sqrt{3})^2$$
$$=1+2+2\sqrt{2}-3$$
$$=2\sqrt{2}.$$

Again, multiplying $2\sqrt{2}$ by $\sqrt{2}$, we get

$$2\sqrt{2}\times\sqrt{2}=4$$

$\therefore$ the rationalising factor is $(1+\sqrt{2}-\sqrt{3})\sqrt{2}$.

Find the rationalising factor of:

53. $1-\sqrt{2}+\sqrt{3}$.
54. $\sqrt{3}+\sqrt{2}-1$.
55. $\sqrt{3}-\sqrt{2}+\sqrt{5}$.
56. $\sqrt{2}+\sqrt{3}-\sqrt{5}$.
57. $3\sqrt{5}-2\sqrt{3}+1$.
58. $3\sqrt{2}+2\sqrt{3}-3\sqrt{6}$.

Example 10. Simplify $\dfrac{1}{2-\sqrt{3}}-\dfrac{1}{\sqrt{3}+\sqrt{2}}+\dfrac{4}{3-\sqrt{5}}$.

$$\text{The expression}=\frac{2+\sqrt{3}}{(2-\sqrt{3})(2+\sqrt{3})}-\frac{\sqrt{3}-\sqrt{2}}{(\sqrt{3}+\sqrt{2})(\sqrt{3}-\sqrt{2})}$$
$$+\frac{4(3+\sqrt{5})}{(3-\sqrt{5})(3+\sqrt{5})}=\frac{2+\sqrt{3}}{4-3}-\frac{\sqrt{3}-\sqrt{2}}{3-2}+\frac{4(3+\sqrt{5})}{9-5}$$
$$=2+\sqrt{3}-\sqrt{3}+\sqrt{2}+3+\sqrt{5}$$
$$=5+\sqrt{2}+\sqrt{5}.$$

Example 11. Simplify $\dfrac{9}{\sqrt{6}+\sqrt{12}+\sqrt{24}-\sqrt{48}-\sqrt{3}}$.

$$\text{The denominator}=\sqrt{6}+\sqrt{12}+\sqrt{24}-\sqrt{48}-\sqrt{3}$$
$$=\sqrt{6}+2\sqrt{3}+2\sqrt{6}-4\sqrt{3}-\sqrt{3}$$
$$=3\sqrt{6}-3\sqrt{3}=3\sqrt{3}(\sqrt{2}-1).$$

$$\therefore \quad \text{the fraction} = \frac{9}{3\sqrt{3}(\sqrt{2}-1)} = \frac{3\sqrt{3}\times\sqrt{3}}{3\sqrt{3}(\sqrt{2}-1)}$$

$$= \frac{\sqrt{3}}{\sqrt{2}-1} = \frac{\sqrt{3}(\sqrt{2}+1)}{(\sqrt{2}-1)(\sqrt{2}+1)}$$

$$= \frac{\sqrt{3}(\sqrt{2}+1)}{2-1} = \sqrt{3}(\sqrt{2}+1).$$

Simplify :

59. $\dfrac{1}{a+\sqrt{a^2-1}} + \dfrac{1}{a-\sqrt{a^2-1}}$.

60. $\dfrac{1}{a-\sqrt{a^2-b^2}} - \dfrac{1}{a+\sqrt{a^2-b^2}}$.

61. $\dfrac{5}{3\sqrt{8}+\sqrt{12}-\sqrt{27}-\sqrt{32}}$.

62. $\dfrac{\sqrt{3}}{\sqrt{5}+1} - \dfrac{\sqrt{5}}{2(\sqrt{3}-1)} + \dfrac{1}{\sqrt{5}-\sqrt{3}}$.

63. $\dfrac{2\sqrt{6}}{\sqrt{2}+\sqrt{3}} + \dfrac{3\sqrt{2}}{\sqrt{3}+\sqrt{6}} - \dfrac{4\sqrt{3}}{\sqrt{6}+\sqrt{2}}$.

64. $\dfrac{a+\sqrt{a^2-1}}{a-\sqrt{a^2-1}} - \dfrac{a-\sqrt{a^2-1}}{a+\sqrt{a^2-1}}$.

65. $\dfrac{\sqrt{x^2+1}+\sqrt{x^2-1}}{\sqrt{x^2+1}-\sqrt{x^2-1}} + \dfrac{\sqrt{x^2+1}-\sqrt{x^2-1}}{\sqrt{x^2+1}+\sqrt{x^2-1}}$.

66. Find the value of $(2+\sqrt{3})^{-3}-(2-\sqrt{3})^{-3}$.

5. Properties of quadratic surds.

Theorem I. *The square root of a quantity cannot be partly rational and partly irrational.*

For, if possible, let

$\sqrt{x}=a+\sqrt{b}$; where $\sqrt{b}$ is a *true surd.*

Squaring both sides, we get

$x=a^2+b+2a\sqrt{b}$.

Hence $\sqrt{b}=\dfrac{x-a^2-b}{2a}$,

i.e., a surd is equal to a rational quantity, which is impossible.

Theorem II. If $x+\sqrt{y}=a+\sqrt{b}$, *where* x *and* a *are both rational and* $\sqrt{y}$ *and* $\sqrt{b}$ *are true surds, then* $x=a$ *and* $y=b$.

For, if x is not equal to a than let x be greater than a and be equal to $a+h$,

$$(a+h)+\sqrt{y}=a+\sqrt{b}$$

$\therefore$ $$h+\sqrt{y}=\sqrt{b}$$

$\therefore$ $$\sqrt{b}=h+\sqrt{y}.$$

or

Thus, the square root of a rational quantity is partly rational and partly irrational, which is impossible by theorem I.

Hence $x=a$ and consequently $\sqrt{y}=\sqrt{b}$.

Ex. Tell the flaw in the following process:

$$3+\sqrt{25}=6+\sqrt{4}$$

Since

$$3=6 \text{ and } 25=4.$$

$\therefore$

NOTE. It is important to remember that theorem II holds good only when $\sqrt{y}$ and $\sqrt{b}$ are *true surds* and not merely in surd form, as in the above example.

The method for finding the square root of expressions of the form $a+\sqrt{b}$, where $\sqrt{b}$ is a true surd, is based upon theorem II and is illustrated below by means of a few examples.

Example 1. Find the square root of $31+4\sqrt{21}$.

Let $\sqrt{(31+4\sqrt{21})}=\sqrt{a}+\sqrt{b}$.

Squaring both sides, we get

$$31+4\sqrt{21}=a+b+2\sqrt{ab}$$

$\therefore$ by theorem II, $a+b=31$ ... ... (i)

and $2\sqrt{ab}=4\sqrt{21}$ ... ... (ii)

$\therefore$ $a-b=\sqrt{(a+b)^2-4ab}$

$=\sqrt{(31)^2-(4\sqrt{21})^2}$

$=\sqrt{625}=25$ ... ... (iii)

By adding (i) and (iii), we get

$$2a=56 \text{ or } a=28$$

and by substituting the value of a in (i), we get

$$b=3.$$

$$\therefore \quad \sqrt{31+4\sqrt{21}} = \sqrt{28}+\sqrt{3}$$
$$= 2\sqrt{7}+\sqrt{3}.$$

Example 2. Find the square root of $21-8\sqrt{5}$.

Let $\sqrt{21-8\sqrt{5}} = \sqrt{a}-\sqrt{b}$.

Squaring both sides, we get

$$21-8\sqrt{5} = a+b-2\sqrt{ab}$$
$$\therefore \quad a+b=21 \quad \ldots \quad \ldots \quad \ldots \quad \text{(i)}$$
$$\text{and} \quad 2\sqrt{ab}=8\sqrt{5} \quad \ldots \quad \ldots \quad \ldots \quad \text{(ii)}$$
$$\therefore \quad a-b = \sqrt{(a+b)^2-4ab}$$
$$= \sqrt{(21)^2-(8\sqrt{5})^2}$$
$$= \sqrt{441-320}$$
$$= \sqrt{121}=11. \quad \ldots \quad \ldots \quad \ldots \quad \text{(iii)}$$

From equations (i) and (iii), we get

$a=16$ and $b=5$

$$\therefore \quad \sqrt{21-8\sqrt{5}} = \sqrt{16}-\sqrt{5}$$
$$= 4-\sqrt{5}.$$

NOTE. In most cases the square root of expressions of the form $a+\sqrt{b}$ can be found by inspection, as illustrated below.

Example 3. Find the square root of $7+2\sqrt{10}$.

Here, we have to find two numbers which when added give 7 and when multiplied give 10.

They are obviously 2 and 5.

Thus, $\sqrt{7+2\sqrt{10}} = \sqrt{5}+\sqrt{2}$.

EXERCISE 85.

Find the square root of:

1. $52+14\sqrt{3}$.
2. $17-12\sqrt{2}$.
3. $41+12\sqrt{5}$.
4. $19+8\sqrt{3}$.
5. $14-3\sqrt{20}$.
6. $94+6\sqrt{245}$.
7. $43-30\sqrt{2}$.
8. $27-10\sqrt{2}$.
9. $87-12\sqrt{42}$.

Find by *inspection* the square root of :—

10. $7+4\sqrt{3}$. 11. $8+2\sqrt{15}$. 12. $5+2\sqrt{6}$.

13. $12+2\sqrt{35}$. 14. $21-8\sqrt{5}$. 15. $8-2\sqrt{7}$.

16. If $\sqrt{(21-6\sqrt{10})}=\sqrt{x}-\sqrt{y}$ find the value of x and y.

Example 4. Find the square root of $\sqrt{32}-\sqrt{24}$.

$$\sqrt{32}-\sqrt{24}=\sqrt{2}(\sqrt{16}-\sqrt{12})$$
$$=\sqrt{2}(4-2\sqrt{3})$$
$$\therefore \sqrt{(32-\sqrt{24})}=\sqrt[4]{2}\sqrt{(4-2\sqrt{3})}.$$

Now proceeding as in example 2, we find that

$$\sqrt{4-2\sqrt{3}}=\sqrt{3}-1$$
$$\therefore \sqrt{(\sqrt{32}-\sqrt{24})}=\sqrt[4]{2}(\sqrt{3}-1).$$

Find the square root of :

17. $\sqrt{18}+\sqrt{10}$. 18. $\sqrt{27}-\sqrt{24}$.

19. $\sqrt{75}+\sqrt{72}$. 20. $\sqrt{128}-\sqrt{120}$.

21. $5\sqrt{3}+2\sqrt{18}$. 22. $6\sqrt{5}-\sqrt{175}$.

Find the fourth root of :

23. $68+48\sqrt{2}$. 24. $7-4\sqrt{3}$.

25. $28-16\sqrt{3}$. 26. $124-32\sqrt{15}$.

Example 5. Simplify $\sqrt{2}(1+\sqrt{3})\sqrt{(2-\sqrt{3})}$.

Extracting the square root of $2-\sqrt{3}$, we find

$$\sqrt{2-\sqrt{3}}=\frac{1}{\sqrt{2}}(\sqrt{3}-1)$$

$$\therefore \text{the expression}=\sqrt{2}(1+\sqrt{3})\frac{1}{\sqrt{2}}(\sqrt{3}-1)=(\sqrt{3}+1)(\sqrt{3}-1)$$
$$=3-1=2.$$

Simplify :

27. $\sqrt{6-\sqrt{(17+12\sqrt{2})}}$. 28. $\dfrac{2-\sqrt{3}}{\sqrt{(7-4\sqrt{3})}}$.

29. $(1+\sqrt{3})(\sqrt{2}-\sqrt{3})\sqrt{(5+2\sqrt{6})}$.

30. $\dfrac{\sqrt{(9+2\sqrt{20})}+\sqrt{(9-2\sqrt{20})}}{\sqrt{(9+2\sqrt{20})}-\sqrt{(9-2\sqrt{20})}}$.

*31. $\dfrac{\sqrt{(5+2\sqrt{6})}+\sqrt{(5-2\sqrt{6})}}{\sqrt{(5+2\sqrt{6})}-\sqrt{(5-2\sqrt{6})}}+\dfrac{\sqrt{(5+2\sqrt{6})}-\sqrt{(5-2\sqrt{6})}}{\sqrt{(5+2\sqrt{6})}+\sqrt{(5-2\sqrt{6})}}$.

*32. $\dfrac{3}{\sqrt{(7+2\sqrt{10})}}+\dfrac{2}{\sqrt{(8-2\sqrt{15})}}-\dfrac{5}{\sqrt{(7+2\sqrt{6})}}$.

*33. Evaluate $\sqrt{x}-\dfrac{1}{\sqrt{x}}$ when $x=3+2\sqrt{2}$.

*34. Find the value of $\dfrac{1+x}{1+\sqrt{(1+x)}}+\dfrac{1-x}{1-\sqrt{(1-x)}}$ when $x=\dfrac{\sqrt{3}}{2}$.

[*Hint.* $1+x=1+\dfrac{\sqrt{3}}{2}=\dfrac{2+\sqrt{3}}{2}$ or $\dfrac{4+2\sqrt{3}}{4}$

and $\sqrt{1+x}=\sqrt{\left(\dfrac{4+2\sqrt{3}}{4}\right)}=\dfrac{\sqrt{3}+1}{2}$.

Similarly, $1-x=\dfrac{2-\sqrt{3}}{2}$ or $\dfrac{4-2\sqrt{3}}{4}$

and $\sqrt{1-x}=\dfrac{\sqrt{3}-1}{2}$.]

***Example 6.** Find the square root of $12+4\sqrt{3}-4\sqrt{5}-2\sqrt{15}$.

Let $\sqrt{(12+4\sqrt{3}-4\sqrt{5}-2\sqrt{15})}=\sqrt{x}+\sqrt{y}-\sqrt{z}$.

Then squaring both sides, we have

$12+4\sqrt{3}-4\sqrt{5}-2\sqrt{15}=x+y+z+2\sqrt{xy}-2\sqrt{xz}-2\sqrt{yz}$

$\therefore \quad x+y+z=12 \quad \ldots \quad \ldots \quad \ldots$ (i)

$2\sqrt{xy}=4\sqrt{3} \quad \ldots \quad \ldots \quad \ldots$ (ii)

$2\sqrt{xz}=4\sqrt{5} \quad \ldots \quad \ldots \quad \ldots$ (iii)

$2\sqrt{yz}=2\sqrt{15} \quad \ldots \quad \ldots \quad \ldots$ (iv)

From (ii) and (iii), we get

$4x\sqrt{yz}=16\sqrt{15} \quad \ldots \quad \ldots \quad \ldots$ (v)

But $2\sqrt{yz}=2\sqrt{15} \quad \therefore \quad 2x=8$, or $x=4$.

From (ii), $y=3$, and from (iii), $z=5$.

Also $x+y+z=4+3+5=12$.

$\therefore \quad \sqrt{(12+4\sqrt{3}-4\sqrt{5}-2\sqrt{15})}=2+\sqrt{3}-\sqrt{5}$.

Find the square root of :

*35. $6+\sqrt{8}+\sqrt{12}+\sqrt{24}$.

*36. $10-\sqrt{24}+2\sqrt{10}-2\sqrt{15}$.

*37. $9+2\sqrt{3}-2\sqrt{5}-2\sqrt{15}$.

*38. $15-4\sqrt{5}-4\sqrt{6}+2\sqrt{30}$.

6. Equations involving surds.

Example 1. Solve $\sqrt{x+16}=\sqrt{x}+2$.

Squaring both sides, we have

$$x+16=x+4+4\sqrt{x}.$$

Hence $4\sqrt{x}=12$

$\therefore$ $\sqrt{x}=3$

$\therefore$ $x=9$.

Example 2. Solve $\sqrt{x+2}+\sqrt{x-3}=\sqrt{4x-3}$.

Squaring both sides, we have

$$x+2+x-3+2\sqrt{(x^2-x-6)}=4x-3.$$

By transposition, $2\sqrt{(x^2-x-6)}=2x-2$

$$=2(x-1)$$

$\therefore$ $\sqrt{(x^2-x-6)}=x-1$.

Squaring both sides, we have

$$x^2-x-6=x^2-2x+1.$$

By transposition, $x=7$.

Example 3. Solve $x+\sqrt{x^2+\sqrt{(4x^2-1)}}=1$.

By transposition, $\sqrt{x^2+\sqrt{(4x^2-1)}}=1-x$.

Squaring both sides, we have

$$x^2+\sqrt{(4x^2-1)}=1-2x+x^2$$

$\therefore$ $\sqrt{(4x^2-1)}=1-2x$.

Squaring both sides, we have

$$4x^2-1=1-4x+4x^2.$$

By transposition, $4x=2$

$\therefore$ $x=\frac{1}{2}$.

EXERCISE 86.

Solve the equations:

1. $\sqrt{3x-2}=4$.
2. $\sqrt{x+1}=3\sqrt{x-1}$.
3. $\sqrt{x-1}=1-\sqrt{x}$.
4. $\sqrt{x+7}=1+\sqrt{x}$.
5. $\sqrt{x+11}=11-\sqrt{x}$.
6. $\sqrt{5(x+2)}=2+\sqrt{5x}$.
7. $\sqrt{3x+4}+4-\sqrt{3x}=0$.
8. $\sqrt{16x^2-25}=3-4x$.
9. $x+a+\sqrt{x^2-a^2}=2a$.
10. $\sqrt{x}+\sqrt{7+x}=\dfrac{21}{\sqrt{7+x}}$.
11. $\sqrt{x}+\sqrt{a+x}=\dfrac{2a}{\sqrt{a+x}}$.
12. $\sqrt[3]{7-4x}=3$.
13. $\sqrt[4]{x-3}=\sqrt[8]{x^2-2x+11}$.
14. $\sqrt[5]{x+4}=\sqrt[10]{x^2+5x+25}$.
15. $\sqrt{x}-\sqrt{4+x}=\dfrac{2}{\sqrt{x}}$.
16. $\sqrt{x}+\sqrt{x-\sqrt{1-x}}=1$.
17. $\sqrt{x+11}-\sqrt{x-4}=3$.
18. $\sqrt{2x+1}+\sqrt{2x-7}=4$.
19. $\sqrt{3x+1}-\sqrt{3x-11}=2$.
20. $\sqrt{5x-1}=1+\sqrt{5x-2}$.
21. $\sqrt{7x-10}+\sqrt{7x+1}=11$.
22. $\sqrt{x^2-3x+5}-\sqrt{x^2-x+1}=1$.
23. $\sqrt{x^2+11x+20}-\sqrt{x^2+5x-1}=3$.
24. $\sqrt{9x+1}=\sqrt{x+1}-\sqrt{4x+1}$.
25. $\sqrt{3x+1}+\sqrt{2(x-6)}=\sqrt{5x+9}$.
26. $\sqrt{x+5}+\sqrt{x+6}=\sqrt{4x+11}$.

Example 4. Solve $\sqrt{4x+9}-\sqrt{4x-7}=2$.

If we subtract $(4x-7)$ from $(4x+9)$, we get

$(4x+9)-(4x-7)\equiv 16$ ... ... [Identity.]

Dividing this identity by the given equation, we get

$$\frac{(4x+9)-(4x-7)}{\sqrt{4x+9}-\sqrt{4x-7}}=\frac{16}{2}=8$$

or $\sqrt{4x+9}+\sqrt{4x-7}=8$ ... [Equivalent equation.]

Adding the given equation and the equivalent equation, we get

$$2\sqrt{4x+9}=10$$

$$\sqrt{4x+9}=5$$

or

$$4x+9=25$$

$$x=4.$$

or

The method illustrated in the solution of this example is called the **Method of identity.**

Ex. State in your own words the steps involved in the method of identity.

Example 5. Solve $\sqrt{2x^2-1}+\sqrt{2x^2-2}=7+4\sqrt{3}$.

$$(2x^2-1)-(2x^2-2)\equiv 1 \ldots\ldots\ldots\ldots [\text{Identity.}]$$

Dividing the identity by the given equation, we have

$$\frac{(2x^2-1)-(2x^2-2)}{\sqrt{2x^2-1}+\sqrt{2x^2-2}}=\frac{1}{7+4\sqrt{3}}=\frac{1(7-4\sqrt{3})}{49-48}$$

$\therefore \sqrt{2x^2-1}-\sqrt{2x^2-2}=7-4\sqrt{3}$.. [Equivalent equation.]

Adding the given equation and the equivalent equation, we get

$$2\sqrt{2x^2-1}=14$$

$\therefore$ $$\sqrt{2x^2-1}=7$$

$\therefore$ $$2x^2-1=49$$

$\therefore$ $$x^2=25$$

$\therefore$ $$x=\pm 5.$$

Solve the equations by the method of identity :

27. $\sqrt{4x+5}-\sqrt{4x-11}=2$.
28. $\sqrt{5x+6}+\sqrt{5x+4}=2$.
29. $\sqrt{5x+6}-\sqrt{5x-14}=2$.
30. $\sqrt{9x-2}-\sqrt{9x-11}=1$.
31. $\sqrt{6x+1}+\sqrt{6x-8}=9$.
32. $\sqrt{7x+8}+\sqrt{7x-20}=14$.
33. $\sqrt{3x^2-11}+\sqrt{3x^2-15}=8+2\sqrt{15}$.
34. $\sqrt{4x^2-23}-2\sqrt{x^2-18}=11-6\sqrt{2}$.
35. $\sqrt{2x+1}-\sqrt{3x+2}=\sqrt{4x+3}-\sqrt{5x+4}$.

[*Hint.* $(2x+1)-(3x+2)=-x-1=(4x+3)-(5x+4)$... identity.]

*Example 6. Solve $\sqrt[3]{1+x}+\sqrt[3]{1-x}=\sqrt[3]{2}$.

Cubing both sides of the equation, we have

$$(1+x)+(1-x)+3\sqrt[3]{(1-x^2)}\{(\sqrt[3]{1+x}+\sqrt[3]{1-x})\}=2.$$

Since it is given that $\sqrt[3]{1+x}+\sqrt[3]{1-x}=\sqrt[3]{2}$, we substitute $\sqrt[3]{2}$ for $\sqrt[3]{1-x}+\sqrt[3]{1-x}$ in the above and get

$$2+3\sqrt[3]{1-x^2}.\sqrt[3]{2}=2$$

$$\therefore \quad 3.\sqrt[3]{2}.\sqrt[3]{1-x^2}=0$$

$$\text{or} \quad \sqrt[3]{1-x^2}=0$$

$$\text{or} \quad 1-x^2=0$$

$$\text{or} \quad x^2=1$$

$$\text{or} \quad x=\pm 1.$$

Note very carefully the artifice used in the solution of this example.

Solve the equations by the above method:

*36. $\sqrt[3]{x+1}-\sqrt[3]{x-1}=\sqrt[3]{2}$.

*37. $\sqrt[3]{(a+\sqrt{x})}+\sqrt[3]{(a-\sqrt{x})}=\sqrt[3]{b}$.

*38. $\sqrt[3]{ax+b}-\sqrt[3]{ax-b}=2b$.

*39. $(a+x)^{\frac{2}{3}}+(a-x)^{\frac{2}{3}}=3(a^2-x^2)^{\frac{1}{3}}$.

*40. $\sqrt[3]{x+13}-\sqrt[3]{x-13}=2$.

Example 7. Solve $\dfrac{x-9}{\sqrt{x}+3}=3-\dfrac{\sqrt{x}-3}{2}$.

$$\frac{x-9}{\sqrt{x}+3}=\frac{(\sqrt{x}+3)(\sqrt{x}-3)}{\sqrt{x}+3}=\sqrt{x}-3.$$

Hence the given equation becomes

$$\therefore \quad \sqrt{x}-3=3-\frac{\sqrt{x}-3}{2}$$

$$\therefore \quad 2\sqrt{x}-6=6-\sqrt{x}+3$$

$$\therefore \quad 3\sqrt{x}=15$$

$$\therefore \quad \sqrt{x}=5$$

$$\therefore \quad x=25.$$

Solve the equations :

*41. $\frac{x-1}{\sqrt{x+1}}=1+\frac{\sqrt{x-1}}{2}$.

*42. $\frac{x-1}{\sqrt{x+1}}=4+\frac{\sqrt{x-1}}{2}$.

*43. $\frac{x-4}{\sqrt{x+2}}=\frac{\sqrt{x-2}}{3}-2$.

*44. $\frac{ax-1}{\sqrt{ax+1}}=4+\frac{\sqrt{ax}-1}{2}$.

*45. $\frac{5x-9}{\sqrt{5x+3}}=1+\frac{\sqrt{5x}-3}{2}$.

*46. $\frac{ax-b^2}{\sqrt{ax+b}}=c+\frac{\sqrt{ax}-b}{c}$.

*47. $\frac{1}{\sqrt{x-\sqrt{x-1}}}+\frac{1}{\sqrt{x-\sqrt{x-2}}}=\frac{3}{2}\sqrt{x}$.

[*Hint.* Rationalise the denominators on the left-hand side.]

*48. $\frac{1}{\sqrt{x-\sqrt{x-2}}}+\frac{1}{\sqrt{x-\sqrt{x+2}}}=1$.

*49. $\frac{1}{1+\sqrt{1-x}}-\frac{1}{1-\sqrt{1+x}}=\frac{1}{x}$.

Example 8. Solve $\frac{\sqrt{x}+\sqrt{3x-1}}{\sqrt{x}-\sqrt{3x-1}}=\frac{5}{3}$.

By componendo-dividendo

$$\frac{(\sqrt{x}+\sqrt{3x-1})+(\sqrt{x}-\sqrt{3x-1})}{(\sqrt{x}+\sqrt{3x-1})-(\sqrt{x}-\sqrt{3x-1})}=\frac{5+3}{5-3}=4$$

$$\frac{2\sqrt{x}}{2\sqrt{3x-1}}=4$$

$\therefore$

$$\sqrt{x}=4\sqrt{3x-1}.$$

$\therefore$ Squaring both sides, we have

$$x=16(3x-1)$$

$$47x=16$$

$$x=\frac{16}{47}.$$

Solve the equations :

50. $\frac{\sqrt{1-x}-1}{\sqrt{1-x}+1}=\frac{1}{3}$.

51. $\frac{\sqrt{x+1}-\sqrt{x-1}}{\sqrt{x+1}+\sqrt{x-1}}=\frac{1}{2}$.

52. $\frac{\sqrt{a+x}+\sqrt{a-x}}{\sqrt{a+x}-\sqrt{a-x}}=4$.

53. $\frac{3x+\sqrt{4x-x^2}}{3x-\sqrt{4x-x^2}}=2$.

*54. $\frac{\sqrt{2x-1}+\sqrt{2x+1}}{\sqrt{2x-1}-\sqrt{2x+1}}=\frac{\sqrt{x+2}+\sqrt{x-2}}{\sqrt{x+2}-\sqrt{x-2}}$.

CHAPTER XIX

ELIMINATION

Let us solve the equations (i) $x+5=9$ (ii) $x+3=8$.

Obviously, the first equation is true when $x=4$ and the second is true when $x=5$, but both of them are never true *simultaneously* for any value of x.

Thus, these two equations are *not consistent* for one and the same value of x.

Now, let us solve the equations (i) $x+a=9$ (ii) $x+b=8$.

The first equation is true when $x=9-a$ and the second is true when $x=8-b$ and both are *simultaneously* true for the same value of x if $9-a=8-b$ or if $a-b=1$.

Thus, these two equations are *consistent* for one and the same value of x if $a-b=1$. The condition discovered here is *general* and helps us in constructing pairs of consistent equations, for example:

$x+3=9$, $x+2=8$ ...here $3-2=1$ and $x=6$,

$x+4=9$, $x+3=8$ ...here $4-3=1$ and $x=5$,

$x+5=9$, $x+4=8$ ...here $5-4=1$ and $x=4$,

and so on.

It is by removing x from the equations $x+a=9$ and $x+b=8$ that we have arrived at the condition $a-b=1$, which is *necessary* for the consistency of these two equations.

The process involved in *throwing off* the unknown from the given equations and arriving at a condition or relation

necessary for their consistency is called **elimination** and the relation thus discovered is called **eliminant.**

NOTE. Since to find the value of n unknown quantities, we require n equations, for their elimination we *generally require one more* equation or in all $n+1$ equations.

An exhaustive treatment of elimination is beyond the scope of this book. Here we shall illustrate by means of examples some important artifices of frequent application.

Example 1. Eliminate x from the equations

$$mx+n=0 \quad \ldots \quad \ldots \quad \ldots \text{ (i)}$$
$$ax^2+bx+c=0 \quad \ldots \quad \ldots \quad \ldots \text{ (ii)}$$

From (i), we get $x=-\frac{n}{m}$.

Substituting $-\frac{n}{m}$ for x in (ii), we get

$$a.\left(-\frac{n}{m}\right)^2+b.\left(-\frac{n}{m}\right)+c=0$$

or $an^2-bmn+cm^2=0$, which is the required eliminant.

Example 2. Eliminate x and y from the equations

$$a_1x+b_1y+c_1=0 \quad \ldots \quad \ldots \quad \ldots \text{ (i)}$$
$$a_2x+b_2y+c_2=0 \quad \ldots \quad \ldots \quad \ldots \text{ (ii)}$$
$$a_3x+b_3y+c_3=0 \quad \ldots \quad \ldots \quad \ldots \text{ (iii)}$$

From (i) and (ii), by the method of cross-multiplication,

$$\frac{x}{b_1c_2-b_2c_1}=\frac{y}{c_1a_2-c_2a_1}=\frac{1}{a_1b_2-a_2b_1}$$

$$\therefore \quad x=\frac{b_1c_2-b_2c_1}{a_1b_2-a_2b_1}, y=\frac{c_1a_2-c_2a_1}{a_1b_2-a_2b_1}.$$

Substituting these values of x and y in (iii), we have

$$a_3.\frac{b_1c_2-b_2c_1}{a_1b_2-a_2b_1}+b_3.\frac{c_1a_2-c_2a_1}{a_1b_2-a_2b_1}+c_3=0,$$

or $a_3(b_1c_2-b_2c_1)-b_3(c_1a_2-c_2a_1)+c_3(a_1b_2-a_2b_1)=0$,

which is the required eliminant.

Example 3. Eliminate x from the equations

$$ax^2+bx+c=0 \quad \ldots \quad \ldots \quad \text{(i)}$$
$$a'x^2+b'x+c'=0 \quad \ldots \quad \ldots \quad \text{(ii)}$$

From (i) and (ii), by the method of cross-multiplication,

$$\frac{x^2}{bc'-b'c}=\frac{x}{ca'-c'a}=\frac{1}{ab'-a'b}$$

$$\therefore \quad \frac{x^2}{bc'-b'c}\times\frac{1}{ab'-a'b}=\left(\frac{x}{ca'-c'a}\right)^2$$

$\therefore \quad (bc'-b'c)(ab'-a'b)=(ca'-c'a)^2$, which is the required eliminant.

Example 4. Eliminate x from the equations

$$ax^3+bx+c=0 \quad \ldots \quad \ldots \quad \ldots \quad \text{(i)}$$
$$x^2+px+q=0 \quad \ldots \quad \ldots \quad \ldots \quad \text{(ii)}$$

Multiplying (ii) by ax and subtracting the resulting equation from (i),

$$-apx^2+(b-aq)x+c=0$$

or
$$apx^2+(aq-b)x-c=0 \quad \text{(iii)}$$

Now, taking (ii) and (iii) and proceeding as in example 3, we find the required eliminant.

EXERCISE 87.

Eliminate x from the equations:

1. $\left.\begin{aligned} x&=2 \\ mx&=n. \end{aligned}\right\}$

2. $\left.\begin{aligned} ax&=b \\ x^2&=m^2. \end{aligned}\right\}$

3. $\left.\begin{aligned} ax+b&=0 \\ a'x+b'&=0. \end{aligned}\right\}$

4. $\left.\begin{aligned} x+m&=0 \\ ax^2+bx+c&=0. \end{aligned}\right\}$

5. $\left.\begin{aligned} ax+b&=0 \\ px^2+qx+r&=0. \end{aligned}\right\}$

6. $\left.\begin{aligned} px+q&=0 \\ ax^3+bx^2+cx+d&=0. \end{aligned}\right\}$

Eliminate t from the equations:

7. $\left.\begin{aligned} x&=at^2 \\ y&=2at. \end{aligned}\right\}$

8. $\left.\begin{aligned} v&=u+ft \\ s&=ut+\tfrac{1}{2}ft^2. \end{aligned}\right\}$

9. Eliminate r from the equations:

$$\left.\begin{aligned} l(r-a)&=m(r-b) \\ l(r-c)&=m(r-d). \end{aligned}\right\}$$

Eliminate x and y from the equations:

10. $\left.\begin{aligned} ax+by&=0 \\ lx^2+mxy+ny^2&=0. \end{aligned}\right\}$

[*Hint.* Take $\frac{x}{y}$ as the variable.]

Eliminate x, y, z from the equations.

11. $\left.\begin{aligned} ax+by+cz&=0 \\ a'x+b'y+c'z&=0 \\ a''x+b''y+c''z&=0. \end{aligned}\right\}$

12. $\left.\begin{aligned} ax+by&=z \\ by+cz&=x \\ cz+ax&=y. \end{aligned}\right\}$

13. $\frac{x}{y-z}=a, \quad \frac{y}{z-x}=b, \quad \frac{z}{x-y}=c.$

14. $\frac{x}{y+z}=a, \quad \frac{y}{z+x}=b, \quad \frac{z}{x+y}=c.$

Eliminate l, m, n from the equations:

15. $\left.\begin{aligned} mz+ny&=l \\ nx+lz&=m \\ ly+mx&=n. \end{aligned}\right\}$

Eliminate t from the equations:

16. $\left.\begin{aligned} t^2+ft+x&=0 \\ t^2+gt+y&=0. \end{aligned}\right\}$

Eliminate l from the equations:

17. $\left.\begin{aligned} xl^2+yl+z&=0 \\ x'l^2+y'l+z'&=0. \end{aligned}\right\}$

Eliminate x from the equations:

18. $\left.\begin{aligned} x^2+d&=0 \\ ax^2+bx+c&=0. \end{aligned}\right\}$

[*Hint.* Put the first equation in the form of $x^2+0x+d=0$.]

19. $\left.\begin{aligned} ax^3+bx+c&=0 \\ a'x^3+b'x+c'&=0. \end{aligned}\right\}$

20. $\left.\begin{aligned} ax^3+bx^2+c&=0 \\ a'x^3+b'x^2+c'&=0. \end{aligned}\right\}$

21. $\left.\begin{aligned} x^2+px+q&=0 \\ x^3+qx+r&=0. \end{aligned}\right\}$

22. $\left.\begin{aligned} ax^2+bx+c&=0 \\ x^3+3x^2+4&=0. \end{aligned}\right\}$

Example 5. Eliminate x from the equations:

$$ax^m - b = 0 \quad \ldots \quad \ldots \quad \text{(i)}$$
$$cx^n - d = 0 \quad \ldots \quad \ldots \quad \text{(ii)}$$

From (i), $x^m = \frac{b}{a}$ and from (ii), $x^n = \frac{d}{c}$,

$\therefore \quad \left(x^m\right)^n = \left(\frac{b}{a}\right)^n$ and $\left(x^n\right)^m = \left(\frac{d}{c}\right)^m$

$\therefore \quad x^{mn} = \left(\frac{b}{a}\right)^n$ and $x^{mn} = \left(\frac{d}{c}\right)^m$

$\therefore \quad \left(\frac{b}{a}\right)^n = \left(\frac{d}{c}\right)^m$, or $a^n d^m = b^n c^m$ is the required eliminant.

Example 6. Eliminate x from the equations $\left.\begin{aligned} ax + \frac{b}{x} &= m \\ ax - \frac{b}{x} &= n. \end{aligned}\right\}$

By adding, we get $2ax = m + n$.

By subtracting, we get $\frac{2b}{x} = m - n$.

Multiplying these results, we get the eliminant

$$4ab = m^2 - n^2.$$

Eliminate x from the equations:

23. $\left.\begin{aligned} ax^2 + b &= 0 \\ cx^3 + d &= 0. \end{aligned}\right\}$ **24.** $\left.\begin{aligned} px^3 - q &= 0 \\ mx^4 - n &= 0. \end{aligned}\right\}$

25. $\left.\begin{aligned} x + \frac{1}{x} &= a \\ x - \frac{1}{x} &= b. \end{aligned}\right\}$ **26.** $\left.\begin{aligned} 2x + \frac{3}{x} &= m \\ 2x - \frac{3}{x} &= n. \end{aligned}\right\}$

27. Eliminate m from the equations:

$$\left.\begin{aligned} \frac{a}{m} + \frac{m}{a} &= p \\ \frac{a}{m} - \frac{m}{a} &= q. \end{aligned}\right\}$$

Example 7. Eliminate x from the equations $\left.\begin{aligned} x^2+\frac{1}{x^2}&=a^2 \\ x+\frac{1}{x}&=b. \end{aligned}\right\}$

Since $x+\frac{1}{x}=b, \quad \therefore \quad \left(x+\frac{1}{x}\right)^2=b^2$

Or $\left(x^2+\frac{1}{x^2}\right)+2=b^2.$

Substituting the value of $x^2+\frac{1}{x^2}$, we get

$$a^2+2=b^2$$

or $a^2-b^2+2=0$, which is the required eliminant.

Example 8. Eliminate x, y from the equations $\left.\begin{aligned} ax+by&=c \\ bx-ay&=d \\ x^2+y^2&=1. \end{aligned}\right\}$

Squaring the first and the second equations and adding the results, we have

$$a^2x^2+b^2y^2+2abxy+b^2x^2+a^2y^2-2abxy=c^2+d^2$$

$$\therefore \quad a^2(x^2+y^2)+b^2(x^2+y^2)=c^2+d^2.$$

Substituting the value of x^2+y^2, we have

$a^2+b^2=c^2+d^2$, which is the required eliminant.

Eliminate x from the equations:

28. $\left.\begin{aligned} x^2+\frac{1}{x^2}&=p^2 \\ x-\frac{1}{x}&=q. \end{aligned}\right\}$

29. $\left.\begin{aligned} x^3+\frac{1}{x^3}&=p^3 \\ x+\frac{1}{x}&=q. \end{aligned}\right\}$

30. $\left.\begin{aligned} x^3-\frac{1}{x^3}&=p^3 \\ x-\frac{1}{x}&=q. \end{aligned}\right\}$

31. $\left.\begin{aligned} x^4+\frac{1}{x^4}&=p^4 \\ x^2+\frac{1}{x^2}&=q^2. \end{aligned}\right\}$

32. $\left.\begin{aligned} x^4+\frac{1}{x^4}&=p^4. \\ x+\frac{1}{x}&=q. \end{aligned}\right\}$

33. $\left.\begin{aligned} x^4+\frac{1}{x^4}&=p^4 \\ x-\frac{1}{x}&=q. \end{aligned}\right\}$

34. $\left.\begin{aligned} x^2-\frac{1}{x^2}&=a^2\\ x-\frac{1}{x}&=b. \end{aligned}\right\}$

Eliminate x, y from the equations:

35. $\left.\begin{aligned} ax+y&=m\\ x-ay&=n\\ x^2+y^2&=1. \end{aligned}\right\}$

36. $\left.\begin{aligned} ax-by&=0\\ bx+ay&=r\\ x^2+y^2&=1. \end{aligned}\right\}$

Example 9. Eliminate x, y from the equations:

$x-y=a$, $x^2-y^2=b^2$ and $x^3-y^3=c^3$.

Dividing the second equation by the first, we get

$$x+y=\frac{b^2}{a}.$$

$\therefore\quad (x+y)^2-(x-y)^2 \quad \text{or} \quad 4xy=\left(\frac{b^2}{a}\right)^2-a^2$

$\therefore\qquad xy=\frac{1}{4}\left(\frac{b^4-a^4}{a^2}\right).$

Also $x^3-y^3=(x-y)^3+3xy(x-y)$

$\therefore\qquad c^3=a^3+\frac{3}{4}\left(\frac{b^4-a^4}{a^2}\right)a$

$\therefore\qquad a^4-4ac^3+3b^4=0$ is the required eliminant.

Eliminate x and y from the equations:

37. $x+y=a$, $x^2+y^2=b^2$, $x^3+y^3=c^3$.

38. $x+y=a$, $x^2-y^2=b^2$, $x^3-y^3=c^3$.

39. $x+y=a$, $x^3+y^3=b^3$, $x^4+y^4=c^4$.

*40. $\left.\begin{aligned} x+y+m&=a\\ xy+xm+ym&=b^2\\ xym&=c^3. \end{aligned}\right\}$

[*Hint.* (i) From the first equation find the value of $x+y$ and from the third the value of xy, and substitute them in the second equation.]

(ii) $(m-x)(m-y)(m-m)\equiv 0$

or $m^3-(x+y+m)m^2+(xy+xm+ym)m-xym\equiv 0.$]

Eliminate x, y, z from the equations:

*41. $\frac{y+z}{y-z}=a, \frac{z+x}{z-x}=b, \frac{x+y}{x-y}=c.$

[*Hint.* Apply. componendo-dividendo and multiply the results.]

*42. $x-y=a, y-z=b, z-x=c.$

*43. Eliminate x from the equations:

$$\frac{a-x}{1+c}=\frac{b}{1+x}=\frac{a+x}{x(1+c)}.$$

[*Hint.* $\frac{(a-x)+(a+x)}{(1+c)+x(1+c)}=\frac{b}{1+x}$.]

*44. Eliminate x and y from the equations:

$$\frac{3x+y}{p}=\frac{x+y}{q}=\frac{3y+x}{r}.$$

[*Hint.* Proceed as in Q. 43.]

45. Eliminate x from the equations:

$$x+x^2=\frac{p}{m}, \quad x^3=\frac{q}{m}.$$

[*Hint.* Add these two equations, take out x common and substitute the value of $x+x^2$.]

Eliminate t from the equations:

*46. $x=a.\ \frac{1+t^2}{1-t^2}, \quad y=b.\ \frac{2t}{1-t^2}.$

[*Hint.* Reduce the equations to the form $\frac{x}{a}=\frac{1+t^2}{1-t^2}$, then add and subtract. Multiply the results.]

47. $x=\frac{2at}{1+t^2}, y=\frac{2bt}{1-t^2}.$

[*Hint.* Add and subtract the values of $\frac{2a}{x}$ and $\frac{2b}{y}$. Multiply the results.]

48 Eliminate m and n from the equations:

$$y=mx+\frac{a}{m},\quad y=nx+\frac{a}{n},\quad mn=-1.$$

[*Hint.* Subtracting the 2nd equation from the first, factorising the results and substituting the value of mn, we get the eliminant.]

*49. Eliminate x and y from the equations:

$$x.y^{a-1}=p,\quad xy^{b-1}=q,\quad xy^{c-1}=r.$$

[*Hint.* Divide the first equation by the 2nd, the 2nd by the 3rd, and proceed as in example 5.]

50. Eliminate x, y, z from the equations:

$x^2(y+z)=a^3$, $y^2(z+x)=b^3$, $z^2(x+y)=c^3$, and $xyz=abc$.

[*Hint.* Multiply the first three equations and substitute $\{x^2(y+z)+y^2(z+x)+z^2(x+y)+2xyz\}$ for $(y+z)(z+x)(x+y)$.]

51. Eliminate x from the equations:

$$x^3+\frac{3}{x}=4(a^3+b^3),\ 3x+\frac{1}{x^3}=4(a^3-b^3).$$

[*Hint.* By adding and subtracting get the values of $\left(x+\frac{1}{x}\right)$ and $\left(x-\frac{1}{x}\right)$, then proceed as in example 6.]

SECTIONAL REVISION V

Test Papers

PAPER 1

1. (i) Prove in full (*a*) $(x^2)^5 = x^{10}$, (*b*) $(3x^2)^4 = 81x^8$.

(ii) Write in shorter form (*a*) 2,700,000,

(*b*) $2{\cdot}3 \times 2{\cdot}3 \times 2{\cdot}3 \times 1{,}000{,}000$.

(iii) Simplify (*a*) $x^0 \times 1$. (*b*) $x^0 \times x$.

2. Find by expansion the approximate value of

(i) $(1{\cdot}0001)^4$. (ii) $(1{\cdot}004)^5$.

3. Expand (i) $(3p+2q)^4$, (ii) $(3p-2q)^4$.

4. Find the values of a and b for which $x^4 - 4x^3 - 2x^2 + ax + b$ may be a perfect square.

5. Find the square root of $57 - 28\sqrt{2}$.

6. Solve the equations
$$\left.\begin{aligned} 3^{x+y+z} &= 9^{x-y+z} \\ 4^{3y+2} &= 16^{2x+z} \\ 2^{2x+3y+z} &= 8^{x+2y}. \end{aligned}\right\}$$

PAPER 2

1. Express the following facts without using the index notation:

(i) the velocity of light is $30{\cdot}02 \times 10^{10}$ *m.m.* per second;

(ii) the distance of a star named L Centuri is $\cdot 26 \times 10^{14}$ miles;

(iii) $30 \times 10^4 \times 2{\cdot}4 \times 10^5 = 72 \times 10^9$.

2. (i) Compare the values of $\left[(3^2)^2\right]^2$ and $3^{2^{2^2}}$.

(ii) Write down the values of $(1)^0$, $(1)^{-4}$, $(-1)^{-3}$, $(-1)^{-2}$.

3. If $a^x = m$, $a^y = n$ and $a^z = (m^y n^x)^z$, shew that $xyz = 1$.

4. Simplify $(1+x)^6 - (1-x)^6$.

5. Find the square root of $31 - 12\sqrt{3}$.

6. Solve the equations (i) $(\sqrt{8})^{x+2} = (\sqrt[5]{8})^{3x+1}$.

(ii) $5^{x-1}.3^{x-2} = 5$.

PAPER 3

1. Express the following facts without using negative indices :

(i) the mass of an atom of hydrogen is $1{\cdot}6\times10^{-24}$ gr.,
(ii) the diameter of an atom of oxygen is 3×10^{-8} cm.;
(iii) the wave length of yellow light is 27×10^{-6} inches.

2. If $a=3$ and $b=2$ write down the values of:
(i) $(ab)^{-1}$, (ii) $(a+b)^{-1}$, (iii) $(a-b)^{-1}$, (iv) $a^{-1}+b^{-1}$.

3. Find the co-efficient of x^3 in the expansion of $(x+1)^5\,(x-2)^4$.

4. Find the square root of $5\sqrt{5}+\sqrt{120}$.

5. If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$, shew that each

$$=\sqrt[3]{\left(\frac{2a^{-3}-5c^{-3}+3e^{-3}}{2b^{-3}-5d^{-3}+3f^{-3}}\right)^{-1}}$$

6. Solve $\sqrt[3]{x+1}-\sqrt[3]{x-1}=\sqrt[3]{2}$.

PAPER 4

1. Write down the values of:
(i) 2^{-3}, (ii) 3^{-2}, (iii) $(\frac{1}{9})^{-2}$ (iv) $(\frac{2}{3})^{-4}$.

2. (*a*) If $16^x=1$, find x.

(*b*) If $10^{\cdot301}=2$ approximately, solve the equations:
(i) $10^x=200$; (ii) $5^x=10$.

3. Rupee one with compound interest @ 5% per annum after 12 years amounts to Rs. $(1+{\cdot}05)^{12}$.

Calculate the value of the amount to 3 places of decimals.

4. Simplify $\dfrac{2+\sqrt{3}}{\sqrt{2+\sqrt{2+\sqrt{3}}}}+\sqrt{\dfrac{2-\sqrt{3}}{2-\sqrt{2-\sqrt{3}}}}$.

5. Solve the equations:
(i) $\sqrt{x+9}=1+\sqrt{x-4}$.
(ii) $\sqrt{x+\sqrt{4x+\sqrt{1+16x}}}=1+\sqrt{x}$.

6. Eliminate x and y from the equations:

$$x+\frac{1}{x}=a, \qquad y+\frac{1}{y}=b, \qquad xy+\frac{1}{xy}=c.$$

PAPER 5

1. If $a=\sqrt{\dfrac{e-1}{e+1}}$, express $\dfrac{1-a}{1+a}$ in terms of e.

2. Simplify (i) $\sqrt{(2+\sqrt{3})}-\sqrt{(2-\sqrt{3})}$.

   (ii) $\dfrac{\sqrt{5}+2}{\sqrt{(9-4\sqrt{5})}}$.

3. Find the square root of $x^2+\dfrac{1}{x^2}+6\left(x-\dfrac{1}{x}\right)+7$.

4. Find the cube root of

   $$x^6+3x^5-3x^4-11x^3+6x^2+12x-8.$$

5. Solve the equations :

   (i) $\sqrt{5x+4}+\sqrt{5x-1}=5$.

   (ii) $\dfrac{ax-1}{\sqrt{ax+1}}=4-\dfrac{\sqrt{ax}-1}{2}$.

6. Eliminate x, y and z from the equations:

   $(y+z)^2=ayz$, $(z+x)^2=bzx$, $(x+y)^2=cxy$.

PAPER 6

1. Write down the values of :

   (i) $(81)^{\frac{3}{4}}$, (ii) $(81)^{1\cdot25}$, (iii) $\sqrt[5]{32^4}$, (iv) $(100)^{-1\cdot5}$.

2. (i) Multiply $x^{\frac{1}{2}}+x^{\frac{1}{4}}y^{\frac{1}{4}}+y^{\frac{1}{2}}$ by $x^{\frac{1}{2}}-x^{\frac{1}{4}}y^{\frac{1}{4}}+y^{\frac{1}{2}}$.

   (ii) Divide $x-y$ by $x^{\frac{1}{4}}-y^{\frac{1}{4}}$.

3. Find the square root of :

   $$9+3x+2y+\tfrac{1}{3}xy+\tfrac{1}{4}x^2+\tfrac{1}{9}y^2.$$

4. Solve the equations (i) $3^{x-1}=3^{x-2}+6$.

   (ii) $\sqrt[3]{1+x}+\sqrt[3]{1-x}=\sqrt[3]{2}$.

5. (i) Rationalise the denominator of $\dfrac{\sqrt{5}-\sqrt{3}-2}{\sqrt{5}+\sqrt{3}+2}$.

   (ii) Simplify $\dfrac{1}{2(\sqrt{5}-\sqrt{3})}+\dfrac{\sqrt{3}}{(\sqrt{5}+1)}-\dfrac{\sqrt{5}}{2(\sqrt{3}-1)}$.

6. Eliminate y from $m=y^x$ and $n=x^y$

PAPER 7

1. (i) One factor of $a+b$ is $a^{\frac{1}{3}}+b^{\frac{1}{3}}$, write down the other.
(ii) Find the continued product of

$$x^{\frac{1}{4}}-y^{\frac{1}{4}},\ x^{\frac{1}{4}}+y^{\frac{1}{4}},\ x^{\frac{1}{2}}+y^{\frac{1}{2}}.$$

2. Expand (i) $(1+x)^6$, (ii) $(1-x)^6$.

3. Find the square root of $x^4+1+2x+x^3+\frac{1}{x^2}+\frac{x^2}{4}$.

4. Solve the equations (i) $\sqrt{x+2}+\sqrt{x-2}=\frac{4}{\sqrt{x+2}}$.

(ii) $\frac{\sqrt{x+3}+\sqrt{x-3}}{\sqrt{x+3}-\sqrt{x-3}}=\frac{1}{2}$.

5. (i) Evaluate $\sqrt[4]{(17-12\sqrt{2})}$ correct up to 3 decimal places.

(ii) Simplify $\dfrac{\left\{(x^m)^{\frac{1}{r}}\quad(x^n)^{-\frac{1}{p}}\right\}^{pr}}{\left\{\sqrt[n]{x^p}\ .\ \sqrt[m]{x^r}\right\}^{mn}}$.

6. Eliminate x from $x^3+\frac{1}{x^3}=a^3$ and $x+\frac{1}{x}=b$.

PAPER 8

1. Draw the graph of 2^x from $x=-2$ to $x=+2$ and read on it the approximate value of $2^{\cdot 5}$.

2. Solve the equations $\left.\begin{aligned}3^{4x-3y}\ .\ 5^{4x}&=5^{3y}\\5^{2x-y+1}\ .\ 6^{2x+1}&=6^y.\end{aligned}\right\}$

3. Expand (i) $(x+a)^5$, (ii) $(x-a)^5$.

4. Find the square root of $4x^4-12x^3+29x^2-30x+25$.

5. Simplify $\dfrac{2^{2n+1}-2^{n+2}+2}{2^{2n+1}-2^{n+1}}$.

6. Eliminate x and y from the equations:

$$a=\frac{1}{x}-x,\ b=\frac{1}{y}-y \text{ and } x^2+y^2=1.$$

*CHAPTER XX

QUADRATIC EQUATIONS, PROBLEMS AND GRAPHS

1. Such statements of equality as

$$3x^2=12$$
$$3x^2-3x=0$$
$$2x(x-2)=(2-x)(1-x)+5$$

are *equations* and not identities because they are not true for all values of x, *e.g.*, none of them is true for $x=7$.

When all the terms are transposed to the left-hand side, these equations read

$$3x^2-12=0$$
$$3x^2-3x=0$$
$$x^2-x-7=0.$$

Each of them involves the square of the variable x but no higher power and is called a **Quadratic Equation** in x.

The general form of a quadratic equation is

$$ax^2+bx+c=0$$

where a, b and c are independent of x.

When there is no term in x, as in $3x^2-12=0$, the equation is called a **pure quadratic**; when there is a term in x, as in $3x^2-3x=0$ or $x^2-x-7=0$, the equation is called an **adfected quadratic.**

Solution of Pure Quadratic

2. Illustrative examples:

Example 1. Solve the equation $7x^2=63$.

Dividing both sides by 7, $x^2=9$.

Taking the square root of each side, we have

$$x=\pm 3.$$

For when $x=+3$ or -3, $7x^2=7\ (\pm 3)^2=63$.

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* This Chapter is not included in the Punjab syllabus.

NOTE. Since the square root of $x^2 = \pm x$, it seems necessary to write the answer as $\pm x = \pm 3$; but on analysis, we find that it leads to

(i) $+x = +3$, (ii) $+x = -3$, (iii) $-x = +3$, (iv) $-x = -3$,

out of which (i) and (iv) give the same solution *viz.*, $x = +3$

and (ii) ,, (iii) ,, ,, $x = -3$.

Example 2. Solve the equation $5(x-3)^2 = 180$.

Dividing both sides by 5, we have

$$(x-3)^2 = 36.$$

Taking the square root of both sides, we have

$$x - 3 = \pm 6$$

i.e. $x - 3 = +6$, $\therefore\ x = +9$

and $x - 3 = -6$, $\therefore\ x = -3$.

$\therefore$ the two roots are $+9$ and -3.

Example 3. Solve the equation $(2x-5)^2 = x^2 - 20x + 73$.

Simplifying the left-hand side, we have

$$4x^2 - 20x + 25 = x^2 - 20x + 73.$$

By transposition, $4x^2 - x^2 = -20x + 73 + 20x - 25$.

$\therefore$ $3x^2 = 48$.

Dividing both sides by 3, $x^2 = 16$.

Taking the square root of both sides,

$$x = \pm 4.$$

Example 4. Solve the equation $\dfrac{1}{1+x} + \dfrac{1}{1-x} = \dfrac{4}{1+x^2}$.

Simplifying the left-hand side, we have

$$\frac{1-x+1+x}{1-x^2} = \frac{4}{1+x^2}$$

or $$\frac{2}{1-x^2} = \frac{4}{1+x^2}.$$

Dividing both sides by 2, we have

$$\frac{1}{1-x^2} = \frac{2}{1+x^2}.$$

By cross-multiplication, we get

$$1 + x^2 = 2 - 2x^2.$$

By transposition, $3x^2=1$

or $x^2=\frac{1}{3}$.

Taking the square root of both sides,

$$x=\pm\frac{1}{\sqrt{3}}.$$

EXERCISE 88.

Solve the following equations:

1. $3x^2+7=55$.
2. $10-2x^2=\frac{1}{2}x^2$.
3. $(x-5)^2=25$.
4. $(x+1)^2=2x+5$.
5. $(x-2)(x-5)=15-7x$.
6. $(2x-3)(x-1)=12-5x$.
7. $\frac{ax^2}{b}=\frac{a^3}{b}$
8. $15x^2-7x^2=20+3x^2$.
9. $8x^2+11=5x^2+14$.
10. $29-10x^2=83-16x^2$.
11. $2x^2+17-5(x^2-2)=3(x^2+5)-12$.
12. $\frac{x^2+7}{x^2-5}=\frac{x^2-15}{x^2-3}$.
13. $\frac{4}{x-3}-\frac{4}{x+3}=\frac{1}{3}$.
14. $\frac{x^2-\frac{1}{x}}{5}+\frac{1-2x^2}{3}=x^2$.
15. $\frac{1}{1+x}+\frac{1}{1-x}=\frac{6}{1+x^2}$.
16. $\frac{1}{a+x}+\frac{1}{a-x}=1$.
17. $12\left(\frac{x}{2}+1\right)^2=108$.

Solution by Factorisation

3. The solution of adfected quadratic equations depends on the principle that if $a\times b=0$, then either $a=0$, or $b=0$.

Example 1. Solve the equation $(x+2)(x-3)=0$.

Since the product of two factors $(x+2)$ and $(x-3)$ is equal to zero, either $x+2=0$, or $x-3=0$.

If $x+2=0$, $x=-2$; if $x-3=0$, $x=+3$.

$\therefore$ the required roots are -2 and $+3$.

For when $x=-2$, we have $(-2+2)(-2-3)=0\times-5=0$

and when $x=3$, " " $(3+2)(3-3)=5\times0=0$.

Example 2. Solve the equation $2x+5=3x^2$.

Reducing it to the standard form $ax^2+bx+c=0$, we have by transposition $3x^2-2x-5=0$.

Factorising $(x+1)(3x-5)=0$

whence $x+1=0$ or $3x-5=0$

$\therefore$ the required roots are -1 and $+\frac{5}{3}$.

Verification. (1) When $x=-1$, $3x^2-2x-5$

$=3.(-1)^2-2(-1)-5$

$=3+2-5=0.$

(2) When $x=\frac{5}{3}$, $3x^2-2x-5$

$=3.(\frac{5}{3})^2-2.\frac{5}{3}-5$

$=\frac{25}{3}-\frac{10}{3}-5=0.$

Example 3. Solve the equation $x-\frac{9}{x}=15\left(1-\frac{3}{x}\right)$.

Multiplying both sides by x, we have

$x^2-9=15x-45.$

By transposition $x^2-15x+36=0.$

Factorising $(x-12)(x-3)=0.$

whence $x-12=0$, or $x-3=0$.

$\therefore$ the required roots are $+12$ and $+3$, which can be verified by substitution.

Example 4. Find the equation whose roots are $+7$ and -5.

The factor corresponding to the root $+7$ is $x-7$,

" " " " " -5 is $x+5$.

Hence $(x-7)(x+5)=0$, or $x^2-2x-35=0$ is the required equation.

EXERCISE 89.

Write down the roots of the following equations:

1. $(x-4)(x-6)=0.$
2. $(x+4)(x+6)=0.$
3. $(x-5)(x+7)=0.$
4. $(x+8)(x-9)=0.$
5. $(1-x)(3-x)=0.$
6. $(2-x)(5+x)=0.$
7. $(x-a)(x+b)=0.$
8. $(2x-3)(3x-4)=0.$
9. $(6x-1)(5x+2)=0.$
10. $(ax-b)(bx+c)=0.$
11. $\left(\frac{x}{3}+2\right)\left(3x-\frac{2}{3}\right)=0.$
12. $x(x+5)=0.$

13. $(x+4)^2=0$.
14. $(3x-7)^2=0$.
15. $(3x+a)^2=0$.
16. $(ax-b)^2=0$.
17. $\{x-(p+q)\}^2=0$.
18. $\{x-(p+q)\}\{x+(p-q)\}=0$.

Solve the following equations:

19. $x^2-9x+14=0$.
20. $x^2+10x+24=0$.
21. $x^2-x-12=0$.
22. $1+7x+12x^2=0$.
23. $27+12x+x^2=0$.
24. $5x^2+4x-1=0$.
25. $6x^2+6=13x$.
26. $3x^2+3=10x$.
27. $x^2-5x=36$.
28. $16x^2-159x=10$.
29. $8x^2+3=14x$.
30. $5(3x^2-4)=44x$.

Frame the equations whose roots are:

31. $7, -3$.
32. ± 4.
33. $+a, -b$.
34. $-12, +15$.
35. $m+n, m-n$.

Note. In solving equations it is important to note the following two principles:

(i) When both sides of an equation are squared or multiplied by an expression containing the variable, new or extraneous roots are introduced into it. For example,

when $x-5=0$, $x=+5$ *only*.

If we take the square of $x=+5$, we get $x^2=+25$.

Taking the square root of both sides, we have

$$x=\pm 5.$$

The original root is $+5$, and -5 is the extraneous root, introduced by squaring both sides of the equation.

Again, if we multiply both sides of equation $x=+5$ by $x-3$, we get

$$x(x-3)=+5(x-3).$$

By transposition, $x(x-3)-5(x-3)=0$,

or $(x-3)(x-5)=0$.

Hence, the two roots are $+3$ and $+5$, out of which $+3$ is extraneous and introduced by multiplying both sides of $x=+5$ by $x-3$.

(ii) If both sides of an equation are divided by an expression containing the variable, we lose some roots, the number of such roots is the same as the degree of that expression. For example, let us suppose

$$(5x+2)(3x-4)=(5x+2)(4x-3).$$

Dividing both sides of the equation by $5x+2$, we have

$$3x-4=4x-3 \text{ or } x=-1.$$

If instead of dividing both sides by $5x+2$, we transpose all the terms to the left-hand side, we get

$$(5x+2)(3x-4)-(5x+2)(4x-3)=0$$

or $$(5x+2)\{(3x-4)-(4x-3)\}=0$$

or $$(5x+2)(-x-1)=0$$

or $$(5x+2)(x+1)=0$$

$\therefore$ either $(5x+2)=0$ and $x=-\frac{2}{5}$,

or $(x+1)=0$ and $x=-1$.

$\therefore$ the two roots are $-\frac{2}{5}$ and -1.

Thus by dividing both sides of the original equation by $5x+2$, we lose the root $x=-\frac{2}{5}$.

Method of Completing the Square

4. The method of completing the square is very important as it enables us to solve quadratic equations which cannot be easily done by the method of factorisation.

The method is illustrated in the following examples.

Example 1. Solve the equation $x^2-3x=18$.

Completing the square by adding to each side the square of half the co-efficient of x, we have

$$x^2-3x+(\tfrac{3}{2})^2=18+(\tfrac{3}{2})^2$$

or $$(x-\tfrac{3}{2})^2=\tfrac{81}{4}.$$

Taking the square root of both sides, we have

$$(x-\tfrac{3}{2})=\pm\tfrac{9}{2}$$

$\therefore$ $$x=\tfrac{3}{2}\pm\tfrac{9}{2}.$$

$\therefore$ the required roots are $+6, -3$.

Example 2. Solve the equation $6x^2+7x-20=0$.

By transposition, $6x^2+7x=20$.

Making the co-efficient of x^2 unity by dividing both sides by 6, we have

$$x^2+\tfrac{7}{6}x=\tfrac{10}{3}.$$

Adding the square of half the co-efficient of x to each side, *i.e.*, $(\frac{7}{12})^2$, we have

$$x^2+\tfrac{7}{6}x+(\tfrac{7}{12})^2=\tfrac{10}{3}+(\tfrac{7}{12})^2$$

$$\therefore \quad (x+\tfrac{7}{12})^2=\tfrac{529}{144}.$$

Taking the square root of both sides, we have

$$x+\tfrac{7}{12}=\pm\tfrac{23}{12}$$

$$\therefore \quad x=-\tfrac{7}{12}\pm\tfrac{23}{12}.$$

Hence the required roots are $-\frac{5}{2}$, $+\frac{4}{3}$.

EXERCISE 90.

Solve the following equations by completing the square :

1. $x^2-4x=45$.
2. $x^2-6x-65=0$.
3. $3x^2+13x=30$.
4. $6x^2-13x+6=0$.
5. $8x^2-14x-9=0$.
6. $15x^2-11x-12=0$.
7. $5x^2-2x-3=0$.
8. $4x^2-48x-25=0$.
9. $3x^2-51x+216=0$.
10. $ax^2-bx-c=0$.

Method of Solving by Formula

5. The method of completing the square, when applied to the general form $ax^2+bx+c=0$, gives us a well-known formula for finding the roots of the quadratic equation.

$$ax^2+bx+c=0.$$

Dividing the equation by a, we get

$$x^2+\frac{b}{a}x+\frac{c}{a}=0.$$

$$\therefore \quad x^2+\frac{b}{a}x=-\frac{c}{a}$$

$$\therefore \quad x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2=\left(\frac{b}{2a}\right)^2-\frac{c}{a}$$

$$\therefore \quad \left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2}$$

$$\therefore \quad x+\frac{b}{2a}=\pm\frac{\sqrt{b^2-4ac}}{2a}$$

$$\therefore \quad x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

This formula should be stated in words and committed to memory.

Example. Solve the equation $5x^2-31x+30=0$ by the formula.

If the given equation is put in the general form $ax^2+bx+c=0$

$$a=+5,\ b=-31,\ c=30, \text{ and}$$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{31\pm\sqrt{(-31)^2-4.\ 5.\ 30}}{2.5}$$

$$=\frac{31\pm\sqrt{961-600}}{10}=\frac{31\pm\sqrt{361}}{10}$$

$$=\frac{31\pm19}{10}=5,\ \frac{6}{5}$$

EXERCISE 91.

Solve the following equations by the formula:

1. $3x^2-5x-2=0.$
2. $3x^2-2x-1=0.$
3. $3x^2-7x+2=0.$
4. $12x^2-17x+6=0.$
5. $4x^2-65x+126=0.$
6. $10x+11=\frac{6}{x}.$
7. $\frac{5}{5-x}+\frac{8}{8-x}=3.$
8. $7x^2+17x+6=0.$
9. $5x^2-23x+12=0.$
10. $15x^2+2px-8p^2=0.$
11. $x^2+px=2(x+p).$
12. $pq(x^2-1)=(p^2-q^2)x.$

6. PROBLEMS INVOLVING QUADRATIC EQUATIONS

Since a quadratic equation has always two roots, the problems involving quadratic equations give us two different solutions. The student has to verify to see which of the two roots satisfies the particular problem.

Generally one of the two solutions will satisfy the given problem and the other solution will satisfy a similar problem, if properly interpreted.

Example 1. Find two consecutive even numbers such that the sum of their squares is 100.

Let x and $x+2$ be the two consecutive even numbers.

$$\therefore \quad x^2+(x+2)^2=100$$
$$\therefore \quad x^2+x^2+4x+4=100$$
$$\therefore \quad 2x^2+4x-96=0$$
$$\therefore \quad x^2+2x-48=0$$
$$\therefore \quad (x+8)(x-6)=0$$

Hence the roots are -8 and $+6$.

Verification. (1) When $x=-8$, $x+2=-6$ and
$$(-8)^2+(-6)^2=100.$$
(2) When $x=+6$, $x+2=+8$ and
$$(+6)^2+(+8)^2=100.$$

Example 2. The perimeter of a rectangle is 54 ft. and its area is 180 sq. ft. Find its length and breadth.

Since the perimeter $=54$ ft., $\therefore$ its semi-perimeter $=27$ ft.

Let the length be x ft. $\therefore$ the breadth $=(27-x)$ ft.,

and the area $=x\,(27-x)$ sq. ft.

$$\therefore \quad x(27-x)=180$$
$$\text{or} \quad x^2-27x+180=0$$
$$\therefore \quad (x-15)(x-12)=0$$

Hence the roots are 15 and 12.

Verification. (1) When the length is 15, breadth $=27-15$ $=12$ and the area $=15\times12=180$.

(2) When the length is 12, breadth $=27-12$ $=15$ and the area $=12\times15=180$.

Example 3. AB is a straight line whose length is a units. Find a point P in it such that $AB.PB=AP^2$.

Let $AP=x$ units, then $PB=(a-x)$ units.



$$\therefore \quad AB.PB=AP^2$$

$$\therefore \quad a(a-x)=x^2$$

or $$x^2+ax-a^2=0$$

$$\therefore \quad x=\frac{-a\pm\sqrt{a^2+4a^2}}{2}=\frac{-a\pm a\sqrt{5}}{2}.$$

Hence the two roots are $\frac{1}{2}(\sqrt{5}-1)a$ and $-\frac{1}{2}(\sqrt{5}+1)a$.

The positive value of x corresponds to the point P of internal division.

The negative value of x corresponds to the point P of external division.

EXERCISE 92.

1. Find a number which is equal to three times its square.

2. The sum of a number and its reciprocal is $\frac{25}{12}$. Find it.

3. The sum of the squares of two consecutive numbers is 365. Find them.

4. The product of two consecutive even numbers exceeds their sum by 142. Find them.

5. The sum of the squares of two consecutive odd numbers is 290. Find them.

6. Two numbers differ by 7 and their reciprocals differ by $\frac{7}{60}$. Find them.

7. The area of a rectangle is 255 sq. ft. If its length be diminished by 1 and breadth increased by 1, it becomes a square. Find its dimensions.

8. A straight line AB is a units in length, find a point P in it such that

(i) $AP^2 = 2\ AB.\ PB.$ (ii) $AP^2 = 2\ PB^2.$

(iii) $AP.\ PB = 3\ PB^2.$ (iv) $2\ AP.\ PB = (\frac{1}{2}\ AB)^2.$

9. The area of a rectangle is 240 sq. ft. and its diagonal is 26 ft. Find its sides.

10. The area of a right-angled triangle is 240 sq. ft. and its hypotenuse is 34 ft. Find its sides.

11. The area of a rectangle is equal to the area of a square whose side is 12 ft. longer than the breadth of the rectangle. If the length of the rectangle be increased by 25 ft. and the breadth diminished by 6 ft., its area remains unaltered. Find its dimensions.

12. A cyclist rode 132 miles in a number of hours which was less by 1 than the number of miles he rode per hour. Find the number of hours he rode.

13. A cyclist rode 75 miles at a uniform rate. If he had gone 3 miles per hour slower, he would have taken one hour and 15 minutes more. What was his rate?

14. A lawn is of rectangular shape. Its length is 24 yds. and breadth 16 yds. A path of uniform width whose area is equal to that of the lawn, goes round it. Find the width of the path.

15. The hypotenuse of a right-angled triangle is less than the sum of the other sides by 8 ft. and its area is 120 sq. ft. Find its sides.

16. If a cyclist had gone 3 miles per hour faster, he would have taken 1 hour and 20 minutes less to ride 80 miles. What time did he take?

17. The circumference of one wheel is 4 ft. more than that of another. If the larger wheel makes 220 revolutions less in a mile than the smaller, find the circumference of each wheel.

18. The front wheel of a carriage makes 60 revolutions more than the hind wheel in going 1,080 ft. If the circumference of the front wheel be increased by 2 ft., it will make 15 revolutions more than the hind wheel. Find the circumference of each wheel.

19. The length, breadth and height of a rectangular room are in the ratio of 8 : 6 : 5; if each of the dimensions be increased by a foot, the area of its four walls would be 1,408 sq. ft. Find the dimensions of the room.

20. A battalion of soldiers is formed into a solid square. If the number of soldiers be reduced by 16, it can be formed into a hollow square 6 deep, having in the front 16 soldiers more than before. What is the number of soldiers?

7. Graphs of Quadratic Functions and Equations.

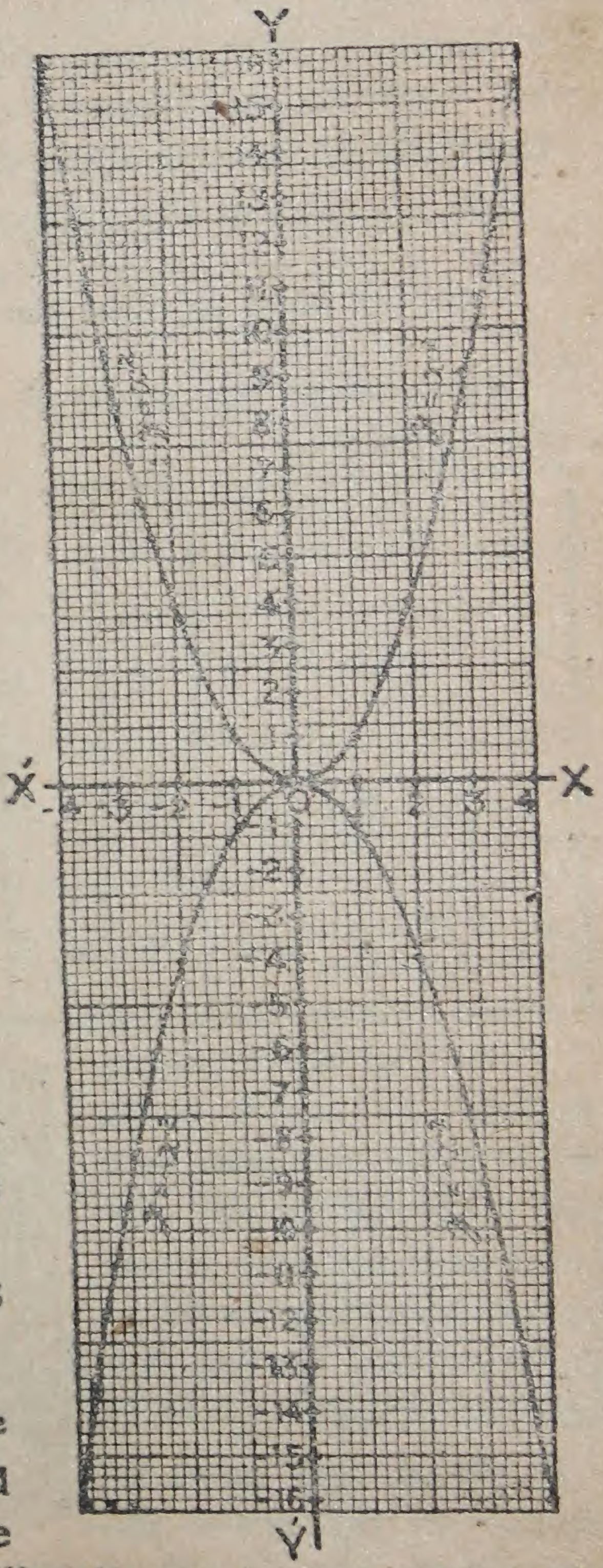
Example 1. Graphs of $y = x^2$ and $y = -x^2$.

Tabulating the values of x and the corresponding values of y in $y = x^2$, we have

$x =$	0	± 1	± 2	± 3	± 4	...
$y =$	0	1	4	9	16	...

If we plot these points to the scale *5mm.* as unit for abscissæ and *4mm.* as unit for ordinates and join these points, we get the annexed upper graph. (p 393).

It is important to note that for any two equal and opposite values of x, the corresponding values of y are equal and *positive*. Consequently the graph lies entirely above the x-axis and is symmetrical with respect to the y-axis.



As x increases indefinitely, y also increases indefinitely but more rapidly than x.

From this graph we can read off approximately the values of squares and square roots. For example, where $x=1{\cdot}8$.

we read $y=3{\cdot}25$ approximately,

$\therefore \quad (1{\cdot}8)^2=3{\cdot}25$ approximately.

Again, at the point where $y=7{\cdot}75$, we read $x=\pm 2{\cdot}8$ approximately.

Hence $\sqrt{7{\cdot}75}=\pm 2{\cdot}8$ approximately.

Draw the graph of $y=-x^2$ with the same origin and axes as for $y=x^2$ and notice that this graph is exactly like the graph of $y=x^2$ but lies just below it and *appears to be its image in x-axis.*

Each of these graphs is a curve known as **parabola.**

Ex. Draw the graph of $y=2x^2$ and show that it is symmetrical with respect to the y-axis, lies throughout above the x-axis and extends up to infinity.

Example 2. Graph of the function $2x^2-5x-3$.

Let y represent the value of the functions, or

$$y=2x^2-5x-3.$$

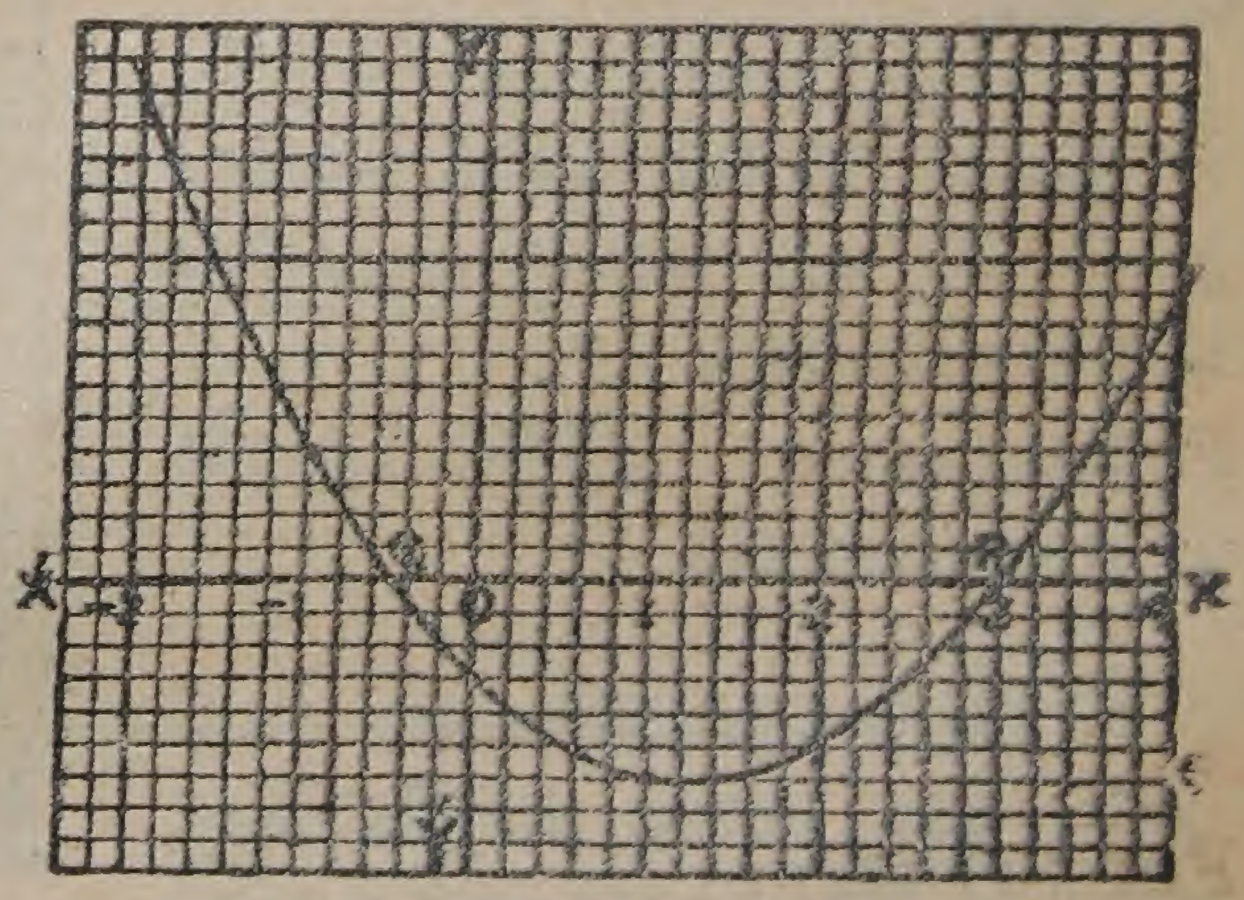
Then we have to trace the graph of the equation

$$y=2x^2-5x-3.$$

We tabulate the values of x and the corresponding values of y as below :

$x=$	0	1	2	3	4	...	−1	−2	...
$2x^2=$	0	2	8	18	32	...	2	8	...
$-5x-3=$	−3	−8	−13	−18	−23	...	2	7	...
y or $2x^2-5x-3=$	−3	−6	− 5	0	9	...	4	15	...

If we plot these points taking 5 small units as unit for abscissæ and one small unit as unit for ordinates and join these points according to their general trend, we get the annexed graph.



In shape, this curve is similar to the curve in example 1, but is differently placed with respect to the axes.

Try to answer the following questions :

1. What are the values of y when $x = 1{\cdot}5,\ 2{\cdot}5,\ -{\cdot}5\ -1{\cdot}5$?
2. For what value of x is the value of y least ?
3. What is the least value of y ?
4. For what value of x is y equal to zero ?
5. Read the co-ordinates of points P and Q where the graph crosses the x-axis.
6. Can you use the graph to solve $2x^2 - 5x - 3 = 0$?
7. For what values of x do we have $2x^2 - 5x - 3 = 6$?

Example 3 Solve graphically the equation

$$2x^2 - 5x - 3 = 0 \quad \ldots \quad (1)$$

First Method. Draw the graph of $y = 2x^2 - 5x - 3$ ... (2) as in example 2.

To solve the equation (1) we have to find the values of x which make $2x^2 - 5x - 3$ equal to zero, *i.e.*, values of x which make $y = 0$.

Now when $y = 0$, the graph of (2) crosses the x-axis at the points P and Q.

The values of x at these points are 3 and $-\frac{1}{2}$.

Hence the roots of (1) are 3 and $-\frac{1}{2}$.

Or, **the roots of equation (1) are the abscissæ of points where the graph of (2) crosses the *x*-axis.**

Second Method. The equation $2x^2 - 5x - 3 = 0$ is the same as the equation $2x^2 = 5x + 3$.

Equate each side of the equation to y,

then $y = 2x^2$ and $y = 5x + 3$.

Draw the graphs of $y = 2x^2$ ... ... (1)

and $y = 5x + 3$ ... ... (2)

For (1) when

$x =$	0	1	2	3	4	...	-1	-2	...
$y =$	0	2	8	18	32	...	2	8	...

For (2) when

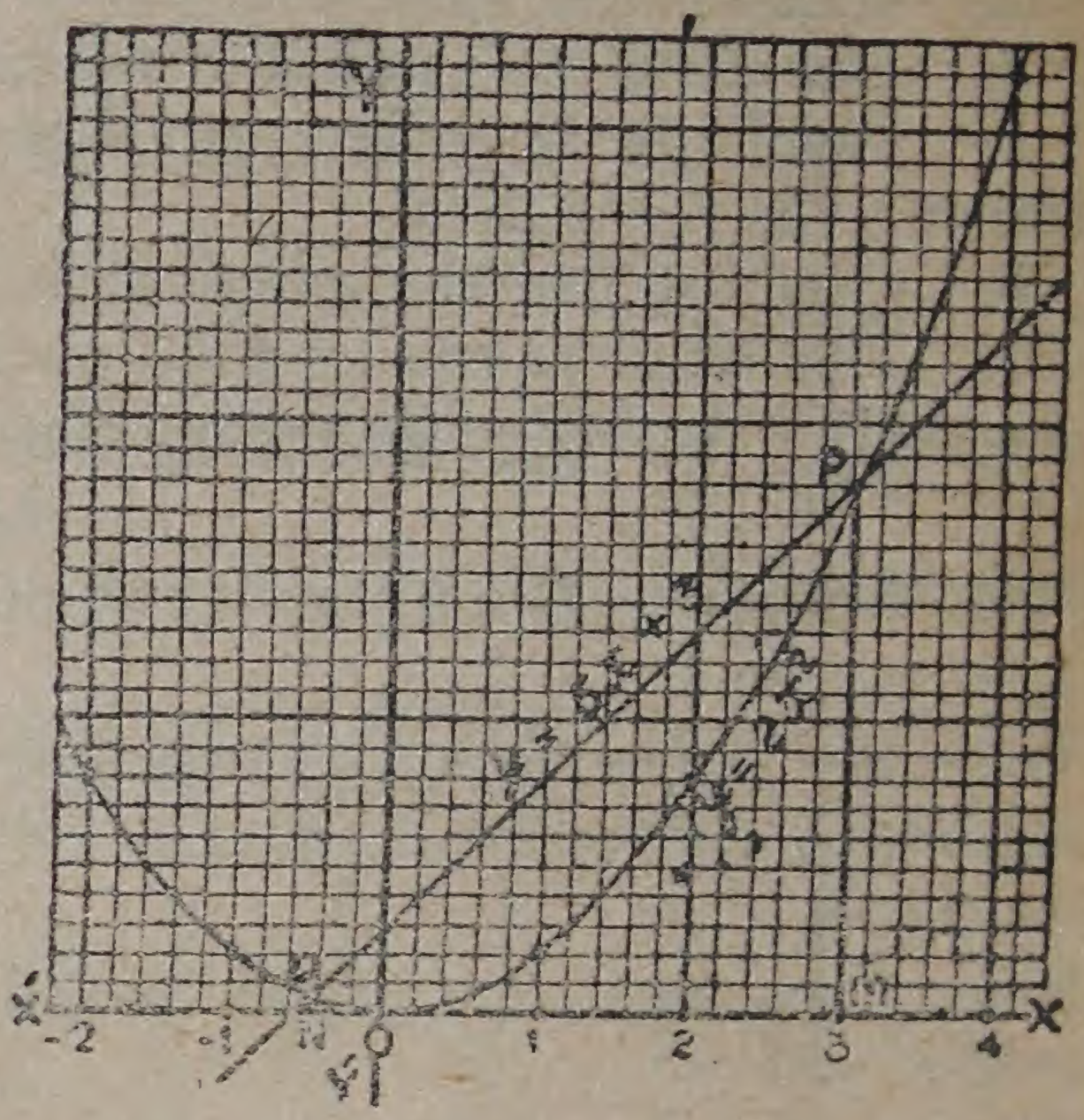
$x =$	0	1	4	...
$y =$	3	8	23	...

These graphs are drawn below to the same scale, taking 5 small units as unit of abscissæ and one small unit as unit of ordinates.

The graphs intersect at points P and Q. The value of x at the point $P = OM = 3$, *i.e.*, when $x = 3$, we have $2x^2 \equiv 5x + 3$.

Similarly, the value of x at $Q = ON = -\frac{1}{2}$, *i.e.*, when $x = -\frac{1}{2}$ we have $2x^2 \equiv 5x + 3$.

Hence the solutions of the equation are 3 and $-\frac{1}{2}$.



In other words, the abscissæ of the points of intersection of the graphs of $y = 2x^2$ and $y = 5x + 3$ are the roots of the equation $2x^2 = 5x + 3$.

Third Method. Another method which is simpler depends upon the use of the graph of $y = x^2$, as illustrated below:

To solve graphically $2x^2 - 5x - 3 = 0$,

let $y = x^2$ ... ... (1)

then $2y - 5x - 3 = 0$ ... ... (2)

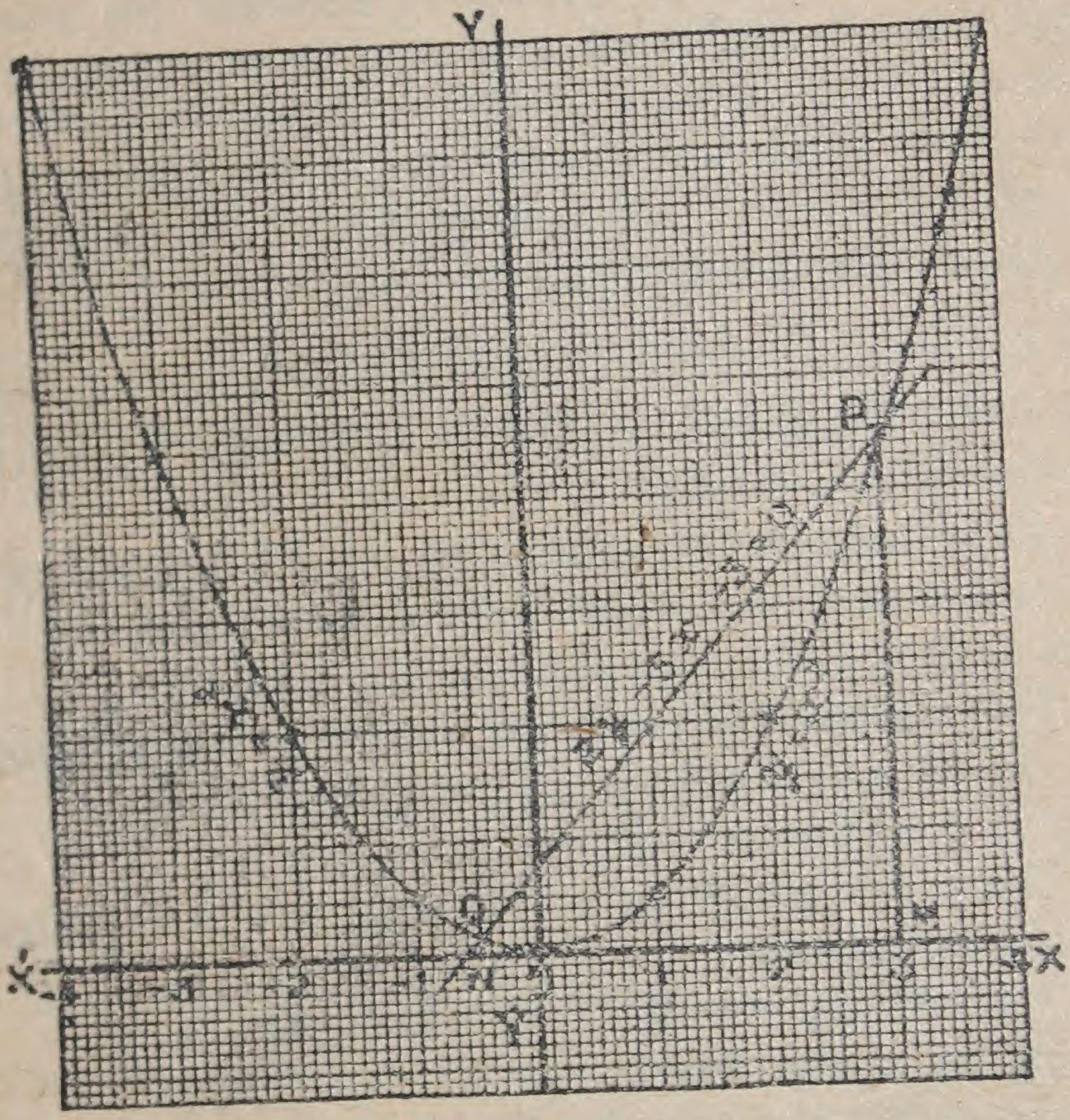
For (1) when

$x =$	0	± 1	± 2	± 3	± 4	...
$y =$	0	1	4	9	16	...

For (2) when

$x =$	1	3	...
$y =$	4	9	...

These graphs are drawn below to the same scale, taking



1 cm. as unit of abscissæ and $\frac{1}{2}$ cm. as unit of ordinates. The graphs intersect at points P and Q.

Measuring the abscissæ of P and Q, we obtain 3 and $-\frac{1}{2}$ as the two roots of the equation.

These three methods being general are applicable in all cases.

The third method is more convenient as it employs the graph of $y=x^2$, no matter what quadratic we wish to solve.

EXERCISE 93.

1. Draw to as large a scale as possible the graph of x^2 between $x=0$ and $x=4$ and read from it the approximate values of

(i) $(1{\cdot}4)^2$, (ii) $(2{\cdot}3)^2$, (iii) $(3{\cdot}2)^2$,
(iv) $\sqrt{3}$, (v) $\sqrt{7{\cdot}3}$, (vi) $\sqrt{13{\cdot}5}$.

2. Draw the graphs of

(i) $y=3x^2$, (ii) $y=4x^2$, (iii) $2y=x^2$, (iv) $y=\frac{x^2}{5}$.

3. Draw the graph of (i) $y^2=x$, (ii) $y^2=-x$ to the same scale and the same axes.

4. Draw neatly the graphs of $y=x^2$ and $y^2=x$ to the same scale and the same axes. Read carefully the co-ordinates of the common points.

5. Solve graphically each of the following equations:

(i) $x^2+x-6=0$. (ii) $x^2-4x+3=0$.
(iii) $3+7x+2x^2=0$.

6. Draw the graph of $y=x^2$ and use it in solving the following equations:

(i) $x^2-5x+6=0$, (ii) $2x^2+5x-3=0$,
(ii) $4x^2+4x-3=0$.

7. Draw the graph of the function $2x^2+5x$ and use it in finding the roots of $2x^2+5x-3=0$.

8. Draw the graphs of $y=x^2$ and $3y+4x-4=0$ to the same scale and the same axes and use them in finding out the roots of $3x^2+4x-4=0$.

9. Draw the graphs of $y=3x^2$ and $10x+8$ to the same scale and the same axes and use them in finding the roots of $3x^2-10x-8=0$.

10. A stone is projected vertically upwards. Its height h ft. above the point of projection, after t seconds is given by $h=96t-16t^2$.

Draw a graph showing the relation between h and t. From the graph, (i) find t when $h=128$, (ii) find h when $t=3$.

CHAPTER XXI

REMAINDER THEOREM AND ITS APPLICATION

1. Remainder Theorem. When an expression in x is divided by another expression of the form $x+p$, ordinarily we find the remainder by actual division. Now, we propose to find the remainder, in such cases, without actual division.

Example 1. Let $2x^2+3x+5$ be the dividend, $x+2$ the divisor, Q the quotient and R the remainder, then, by the identity,

$$\text{dividend} \equiv \text{divisor} \times \text{quotient} + \text{remainder},$$

$$(x+2)\times Q+R \equiv 2x^2+3x+5 \quad \ldots \quad \ldots \quad \text{(i)}$$

Since (i) is an identity and is true for *any value* of x, therefore it is true when $x=-2$.

Thus, $(-2+2)\times Q+R=2(-2)^2+3(-2)+5$

or $0\times Q+R=8-6+5$

or $R=7.$

[Verify the result by actual division.]

Example 2. Let $2x^2+3x+5$ be the dividend, $x-2$ the divisor, Q the quotient and R the remainder,

then, $(x-2)\times Q+R \equiv 2x^2+3x+5 \quad \ldots \quad \text{(ii)}$

As (ii) is true for *any value* of x, therefore it is true when $x=2$.

Thus, $(2-2)\times Q+R=2(2)^2+3(2)+5$

or $0\times Q+R=8+6+5$

or $R=19.$

[Verify the result by actual division.]

Example 3. Let ax^3+bx^2+cx+d be the dividend, $x-p$ the divisor, Q the quotient and R the remainder,

then, $(x-p)Q+R\equiv ax^3+bx^2+cx+d$ ... (iii)

Since (iii) is true for all values of x, therefore it is true when $x=p$.

Thus, $(p-p)Q+R=a(p)^3+b(p)^2+c(p)+d,$

or $0+R=ap^3+bp^2+cp+d,$

or $R=ap^3+bp^2+cp+d.$

[Verify the result by actual division.]

If the above dividend is to be divided by $x+p$,

then $(x+p)Q+R\equiv ax^3+bx^2+cx+d$ ... (iv)

Since (iv) is true for all values of x, therefore it is true when $x=-p$.

Thus, $(-p+p)Q+R=a(-p)^3+b(-p)^2+c(-p)+d.$

or $0+R=a(-p)^3+b(-p)^2+c(-p)+d.$

[Simplify this result and verify it by actual division.]

From these examples we deduce the following theorem:

Theorem. *When an expression in x is the dividend and $x\pm p$ the divisor; the remainder R is obtained by substituting $\mp p$ for x in the expression.*

This is known as the **Remainder Theorem.**

Note. It is useful to note that in (i) if $x+2=0$ we get R, similarly, if in (ii), (iii) and (iv) we put respectively $x-2=0$, $x-p=0$ and $x+p=0$, we get R.

2. Factor Theorem.

Example 4. Prove that $x-1$, $x+1$, $x-2$ are the factors of $2x^4-5x^3+5x-2$.

(i) When $x-1$ is the divisor,

$$R=2(+1)^4-5(+1)^3+5(+1)-2$$
$$=2-5+5-2=0.$$

Since in this case the remainder is 0,

$\therefore$ $x-1$ is a factor of the given expression.

(ii) When $x+1$ is the divisor,

$$R=2(-1)^4-5(-1)^3+5(-1)-2$$
$$=2+5-5-2=0.$$

Since in this case the remainder is 0,

$\therefore$ $x+1$ is a factor of the given expression.

(iii) When $x-2$ is the divisor

$$R=2(+2)^4-5(+2)^3+5(+2)-2$$
$$=32-40+10-2=0.$$

Since in this case the remainder is 0,

$\therefore$ $x+2$ is a factor of the given expression.

We see from these examples that

if an expression in x is reduced to zero on substituting $(\mp p)$ *for x, then* $(x \pm p)$ *is a factor of the expression.*

This is known as the **Factor Theorem.**

In several cases, the two rules deduced from this theorem, as given below, will be found useful :—

Rule 1. $x-1$ *is a factor of an expression in x, if the sum of the co-efficients of terms containing x plus the term containing no x be equal to zero.*

Take for instance, $ax^4+bx^3+cx^2+dx+e$.

The remainder when it is divided by $x-1$ is

$$a.1^4+b.1^3+c.1^2+d.1+e=a+b+c+d+e$$
$$=\text{the sum of the co-efficients.}$$

If this is zero, then $x-1$ is a factor of $ax^4+bx^3+cx^2+dx+e$.

Rule 2. $x+1$ *is a factor of an expression, if the sums of the co-efficients of the alternate terms be equal, provided the terms are properly arranged and no term is missing.*

[Term containing no x is supposed to contain x^0.]

Take for instance, $ax^4+bx^3+cx^2+dx+e$.

The remainder when it is divided by $x+1$ is

$a.(-1)^4+b.(-1)^3+c.(-1)^2+d.(-1)+e=a-b+c-d+e$.

This will be zero, if $a+c+e=b+d$;

i.e. if the sum of the co-efficients of 1st, third and fifth terms = the sum of the co-efficients of 2nd and fourth terms.

Example 5. Resolve into factors $x^3-6x^2+12x-7$.

The sum of the co-efficients $=1-6+12-7=0$;

$\therefore$ by rule 1, $x-1$ is a factor of the expression.

Proceeding as in Type XII,

$$x^3-6x^2+12x-7=x^2(x-1)-5x(x-1)+7(x-1)$$
$$=(x-1)(x^2-5x+7).$$

Example 6. Resolve into factors $x^3+8x^2+19x+12$.

Here, the sum of the co-efficients of the 1st and third terms $=1+19=20$, and the sum of the co-efficients of the 2nd and the fourth terms $=8+12=20$;

$\therefore$ by rule 2, $x+1$ is a factor of the expression.

Proceeding as in Type XII,

$$x^3+8x^2+19x+12=x^2(x+1)+7x(x+1)+12(x+1)$$
$$=(x+1)(x^2+7x+12)$$

As $x^2+7x+12$ can be factorised into $(x+3)(x+4)$,

$\therefore$ $x^3+8x^2+19x+12=(x+1)(x+3)(x+4)$.

Example 7. Resolve into factors $x^5+x^3-3x^2-2x+3$.

When the missing term in the form of $0.x^4$ is added, the expression $=x^5+0.x^4+x^3-3x^2-2x+3$ and the sum of the co-efficients of the 1st, 3rd and 5th terms $=1+1-2=0$,

whereas the sum of the co-efficients of the 2nd, 4th and 6th terms $=0-3+3=0$.

As these sums are equal, $x+1$ is a factor of the expression.

Proceeding as in Type XII,

$$x^5+x^3-3x^2-2x+3=x^4(x+1)-x^3(x+1)+2x^2(x+1)$$
$$-5x(x+1)+3(x+1)$$
$$=(x+1)(x^4-x^3+2x^2-5x+3).$$

Again, as the sum of the co-efficients of $x^4-x^3+2x^2-5x+3$ is 0,

$\therefore$ $x-1$ is a factor of $x^4-x^3+2x^2-5x+3$.

$$\therefore\ x^4-x^3+2x^2-5x+3=x^3(x-1)+2x(x-1)-3(x-1)$$
$$=(x-1)(x^3+2x-3).$$

Again, as the sum of the co-efficients of x^3+2x-3 is 0.

$\therefore$ $x-1$ is a factor of x^3+2x-3.

$$\therefore\ x^3+2x-3=x^2(x-1)+x(x-1)+3(x-1)$$
$$=(x-1)(x^2+x+3).$$

Since x^2+x+3 cannot be factorised

$$\therefore\ x^5+x^3-3x^2-2x+3=(x+1)(x-1)(x-1)(x^2+x+3).$$

EXERCISE 94.

By the Remainder Theorem, find the remainder when

1. x^3-5x^2+6x-3 is divided by $x-1$.
2. $2x^3-3x^2+4x-1$ is divided by $x+2$.
3. $4x^3-2x^2+5x-3$ is divided by $x-a$.
4. $5x^3-x^2+3x-4$ is divided by $x+m$.
5. $2x^4+7x^3a-x^2a^2+11xa^3-6a^4$ is divided by $x+2a$.

Prove that:

6. $x-1$ is a factor of $x^4-3x^3+x^2+3x-2$.
7. $x+2$ and $x-3$ are the factors of $x^3-3x^2-4x+12$.
8. $x-2$ and $x+4$ are the factors of $2x^3+7x^2-10x-24$.
9. $(a+b)$ and $(b+c)$ are the factors of $(a+b+c)^3-a^3-b^3-c^3$.

[*Hint.* Substitute in the expression $-b$ for a in the first case and $-c$ for b in the second case.]

10. a is a factor of $(a+b+c)^3-(b+c-a)^3-(c+a-b)^3-(b+a-c)^3$. [*Hint.* $a=(a-0)$.]

Resolve into factors using the above two rules:

11. x^3+x^2+x-3.

12. $x^3-6x^2+11x-6$.

13. $x^3+9x^2+23x+15$.

14. $3x^3+x^2+x-5$.

15. $4x^3+3x^2+x+2$.

16. $2x^4+3x^3+2x^2-3x-4$.

17. $x^4+2x^3-5x^2+2$.

18. x^4-x^3+7x+5.

19. $x^4+x^3-7x^2-x+6$.

20. $2x^4-x^3-8x^2+11x-4$.

Example 8. For what value of m will $x-2$ be a factor of x^3-5x^2+mx+m?

$x-2$ will be a factor of the given expression

if $(2)^3-5(2)^2+m(2)+m=0$

or if $-12+3m=0$

or if $m=4$.

21. For what value of p is $x^2-(p+2)x+6$ exactly divisible by $x-p$?

22. Find the condition that ax^3+bx^2+cx+d may be exactly divisible by $x-1$.

23. Find the value of a, b for which x^2+x-6 may be a factor of x^3-ax^2-bx-6.

24. What number must be added to $5x^4-7x^3+6$ so that the result may be divisible by $x+3$?

25. If $x^2-4ax-19a^2-b$ is exactly divisible by $x-7a$, prove that $b=2a^2$.

26. The expression px^3+2x^2-3 and x^3-px+4 leave the same remainder when they are divided by $x-2$. Find the value of p.

Let us consider the relation between the algebraic sum of the co-efficients of a given expression and that of the co-efficients of its factors.

Taking the continued product of any number of binomial expressions, say $(2x+3)$, $(7x-4)$, $(3x-1)$, we get

$$42x^3+25x^2-49x+12\equiv(2x+3)(7x-4)(3x-1).$$

Since it is an identity, it is true for all values of x and therefore it is true when $x=1$.

Putting $x=1$ in the identity, we get the algebraic sum of the co-efficients of the expressions and the algebraic sum of the co-efficients of each of its factors as shown below:

$$42.(1)^3+25.(1)^2-49(1)+12=(2.1+3)(7.1-4)(3.1-1)$$

or $$42+25-49+12=(2+3)(7-4)(3-1).$$

Obviously $(42+25-49+12)$ or 30 is exactly divisible by each of $(2+3)$, $(7-4)$, $(3-1)$, *i.e.*, the algebraic sum of the co-efficients of the given expression is exactly divisible by the algebraic sum of the co-efficients of each of its factors.

Thus, we deduce the following rule :—

Rule 3. *The algebraic sum of the co-efficients of each factor of an expression is a factor of the algebraic sum of its co-efficients.*

Obviously the converse of this rule does not hold good for, in above example if we reverse the order of the co-efficients in the binomials we get $(3x+2)$, $(4x-7)$ and $(1-3x)$; the sum of the co-efficients of each of these is a factor of the sum of the co-efficients of the original expression but none of them is its factor.

Therefore rule 3 is a sure test of the rejection of impossible factors and not a sure test of the selection of the real factors. Thus its importance lies in the simplification of the process of selection, as illustrated in the next two examples.

Example 9. Factorise x^3-5x^2-5x-6.

As the expression is of the 3rd degree, it can have either three linear factors or one linear and one quadratic; *i.e., it must have a linear factor.*

As the first term of the expression is x^3, the first term of the linear factor is x.

As the last term of the expression is -6, the last term of the linear factor may be one of

$$+1,\ -1,\ +2,\ -2,\ +3,\ -3,\ +6,\ -6.$$

Or the linear trial factors could be

$(x+1)$, $(x-1)$, $(x+2)$, $(x-2)$, $(x+3)$, $(x-3)$, $(x+6)$ or $(x-6)$.

Since, in this case, neither rule 1 holds good nor rule 2, therefore $(x+1)$ or $(x-1)$ cannot be a factor of the expression,

Again, the sum of the co-efficients of the expression is

$$1-5-5-6=-15,$$

whereas the sum of the co-efficients of $x+2$ is 3 and it is a factor of 15, therefore $x+2$ may be a factor of the expression : the sum of the co-efficients of $x-2$ is -1 and it is a factor of 15, therefore $x-2$ may be a factor of the expression ; the sum of the co-efficients of $x+3$, $x-3$, $x+6$ is 4, -2, 7 respectively and none of them is a factor of 15, therefore none of $x+3$, $x-3$, $x+6$ can be a factor of the expression ; the sum of the co-efficients of $x-6$ is -5 and it is a factor of 15, thefore $x-6$ may be a factor of the expression.

Hence $x+2$, $x-2$, $x-6$ are the only binomials left for trial.

Substituting -2 or $+2$ for x in the given expression, we find that it does not vanish, but on substituting $+6$ for x, we see that it vanishes.

$\therefore$ $x-6$ is one of the factors.

The given expression can be put as

$$x^2(x-6)+x(x-6)+(x-6)=(x-6)(x^2+x+1).$$

As x^2+x+1 cannot be further factorised,

$\therefore$ $(x-6)(x^2+x+1)$ are the required factors.

Example 10. Factorise $3x^4+4x^3-57x^2-18x-40$.

Since, in this case, rules 1, 2 do not hold good,

$\therefore$ $(x-1)$ or $(x+1)$ is not a factor of this expression.

As the last term of the expression is -40, the last term of the linear factor (if any) would be 2, 4, 5, 8, 10 or 20.

Hence the possible trial factors are
$x+2$, $x-2$, $x+4$, $x-4$, $x+5$, $x-5$, $x+8$, $x-8$, $x+10$, $x-10$, $x+20$, $x-20$.

The sum of the co-efficients of the expression
$=3+4-57-18-40=-108$.

The sum of the co-efficients of the trial factors is respectively 3, -1, $+5$, -3, $+6$, -4, $+9$, -7, $+11$, -9, $+21$ and -19.

As 5, 9, -7, 11, 21 and -19 are not the factors of 108, we reject $x+4$, $x+8$, $x-8$, $x+10$, $x+20$ and $x-20$.

On substituting -2 or $+2$ for x in the given expression, we find that it does not vanish, therefore we reject $x+2$ and $x-2$ as well.

On substituting $+4$ for x in the given expression, we find that it vanishes.

$\therefore$ $x-4$ is one of its factors.

Hence $3x^4+4x^3-57x^2-18x-40$
$=3x^3(x-4)+16x^2(x-4)+7x(x-4)+10(x-4)$
$=(x-4)(3x^3+16x^2+7x+10)$.

If we substitute -5 for x in $3x^3+16x^2+7x+10$, it vanishes.

$\therefore$ $3x^3+16x^2+7x+10=3x^2(x+5)+x(x+5)+2(x+5)$
$=(x+5)(3x^2+x+2)$

Since $3x^2+x+2$ cannot be further factorised
$\therefore$ $(x-4)(x+5)(3x^2+x+2)$ are the required factors.

Factorise :

27. $x^3+3x^2-10x-24$.
28. $x^3-5x^2-2x+24$.
29. $6x^3+x^2-19x+6$.
30. $x^3-19x-30$.
31. $x^3-39x+70$.
32. x^3-x^2+12.
33. $x^3-5x+12$.
34. $x^4-2x^3-20x^2+39x+36$

35. Prove that $(b+c)(c+a)(a+b)$ is a factor of $(a+b+c)^n-a^n-b^n-c^n$ and $(b-c)(c-a)(a-b)$ is a factor of $(b-c)^n+(c-a)^n+(a-b)^n$, where n is an odd number.

[*Hint.* Substitute $-c$ for b and put $b+c=0$.]

3. Divisibility of $x^n \pm a^n$ by $x \pm a$.

(i) When x^n-a^n is divided by $x-a$, the remainder is $a^n-a^n=0$,

$\therefore$ x^n-a^n is *always* divisible by $x-a$

(ii) When x^n-a^n is divided by $x+a$, the remainder is $(-a)^n-a^n$ and is equal to 0 or $-2a^n$, according as n is even or odd.

$\therefore$ x^n-a^n is divisible by $x+a$ only when n is *even*.

(iii) When x^n+a^n is divided by $x+a$, the remainder is $(-a)^n+a^n$ and is equal to 0 or $2a^n$, according as n is *odd* or *even*.

$\therefore$ x^n+a^n is divisible by $x+a$ only when n is *odd*.

(iv) When x^n+a^n is divided by $x-a$, the remainder is $a^n+a^n=2a^n$.

$\therefore$ x^n+a^n is *never* divisible by $x-a$.

Example 1. Shew that 10^n-1 is *always* divisible by 9.

Since x^n-a^n is always divisible by $x-a$,

$\therefore$ by putting $x=10$ and $a=1$, we find that

10^n-1^n is always divisible by $10-1$,

or 10^n-1 is always divisible by 9.

Example 2. Simplify $\dfrac{8x^3+27y^3}{2x+3y}$.

Since x^n+a^n is divisible by $x+a$ when n is *odd*,

$$\therefore \quad \frac{8x^3+27y^3}{2x+3y}=\frac{(2x)^3+(3y)^3}{2x+3y}=(2x)^2-(2x)(3y)+(3y)^2$$

$$=4x^2-6xy+9y^2.$$

Example 3. Simplify $\dfrac{x^5+32y^5}{x+2y}$.

Since x^n+a^n is divisible by $x+a$ when n is *odd*.

$$\therefore \quad \frac{x^5+32y^5}{x+2y}=\frac{(x)^5+(2y)^5}{x+2y}$$

$$=x^4-x^3(2y)+x^2(2y)^2-x(2y)^3+(2y)^4$$

$$=x^4-2x^3y+4x^2y^2-8xy^3+16y^4.$$

EXERCISE 95.

Shew that

1. 9^n-1 is *always* divisible by 8.
2. 10^n+1 is *never* divisible by 9.
3. 10^n-1 is divisible by 11 when n is *even*.
4. 10^n+1 is divisible by 11 when n is *odd*.
5. $4^{3n}-3^{3n}$ is *always* divisible by 37.
6. $8^{2m}-1$ is *always* divisible by 63.
7. 17^8-5^8 is divisible by 6 and 11.

Write down by *inspection* the quotient of :

8. $(8a^3-x^3)\div(2a-x)$.
9. $(a^3+64b^3)\div(a+4b)$.
10. $(64a^6-x^6)\div(2a+x)$.
11. $(27m^3-1)\div(3m-1)$.
12. $(x^4-256y^4)\div(x-4y)$.
13. $(1-16m^4)\div(1+2m)$.
14. $(32a^5+b^5)\div(2a+b)$.

*CHAPTER XXII

INTEGRAL FUNCTION, HOMOGENEITY, SYMMETRY, CYCLIC ORDER, INDETERMI NATE CO-EFFICIENTS

1. Variables and constants. Let us consider the variation in the value of:

$2p^2+3p+5$ ... ... (i)

and $3x^2-4x+6$ ... ... (ii)

If in expression (i) $p=1$, its value $=10$

if " " " $p=2$, " " $=19$

if " " " $p=3$, " " $=32$.

Thus, the value of expression (i) *varies as the value of* p *varies*—p is the **variable in this expression.**

Similarly, the value of expression (ii) *varies as the value of* x *varies*—x is the **variable.**

In each of these expressions, *only one letter occurs.*

Now, let us consider the expression $a^2x^2+abx+b^2$.

Here, three letters a, b, x occur. If the value of this expression is supposed to depend upon the value of a only, then a is the variable and the letters x, b are non-variables or **constants.** If its value is supposed to depend upon x or b, then x or b is the variable and the remaining letters are the constants.

It is usual to take x, y, z, as variables and a, b, c as constants.

2. Function. An algebraical expression which contains x as the variable is called a **function of** x, and is denoted as $f(x)$, $F(x)$, $\phi(x)$, &c. If it contains x and y both as the variables, it is called a function of x and y and is denoted as $f(x,y)$, $F(x,y)$, etc.

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* This chapter is meant for the brighter section of the class, as an extra assignment.

For instance, x^3+4x^2-5x+7 is a function of x and may be denoted by $f(x)$ and $3x^2+4xy+2y^2$ is a function of x and y and may be denoted by $f(x, y)$.

A function is said to be **integral** when none of the variables appears in any denominator, and it is said to be **fractional** if the variables occur in its denominators. Thus x, y, z being the variables and a, b, c, d, &c., being the constants,

$ax^2+by^2+cz^2+dxy+exz+fyz$

and $\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}+\frac{xy}{ab}+\frac{xz}{bc}+\frac{yz}{ac}$ are integral functions,

and $\frac{a}{x^2}+bx+c$ and $ax^3+\frac{b}{y^3}+cz^3+dxyz$ are fractional functions.

A function is said to be **rational** if it contains no roots of the variables and **irrational** if it contains roots of the variables; for instance, $ax^2+2bxy+cy^2$ is rational and $ax+2b\sqrt{xy+cy}$ is irrational.

In this chapter we shall consider only the rational integral functions.

It is important to note that if in a particular discussion $f(x)$ stands for a particular expression, say, $3x^2-4x+5$, then $f(y)$ would stand for $3y^2-4y+5$,

$f(p)$	,,	,,	,,	$3p^2-4p+5$,
$f(-m)$	,,	,,	,,	$3(-m)^2-4(-m)+5$,
$f(1)$	,,	,,	,,	$3.1^2-4.1+5$,
$f(0)$	,,	,,	,,	$30.^2-4.0+5$,
$f(p+1)$	,,	,,	,,	$3(p+1)^2-4(p+1)+5$.

Example 1. If $f(x)=x^3-3x^2+3x-1$, find the value of (i) $f(-x)$, (ii) $f(p-1)$.

(i) Putting $-x$ for x in the given expression, we have

$f(-x)=(-x)^3-3(-x)^2+3(-x)-1.$

$=-x^3-3x^2-3x-1.$

(ii) Putting $(p-1)$ for x in the given expression, we have

$$f(p-1)=(p-1)^3-3(p-1)^2+3(p-1)-1$$
$$=p^3-3p^2+3p-1-3(p^2-2p+1)+3p-3-1$$
$$=p^3-3p^2+3p-1-3p^2+6p-3+3p-3-1$$
$$=p^3-6p^2+12p-8.$$

3. Degree of Function. The degree or dimension of an integral function is the same as the degree of the highest term in it.

Thus, the degree of $2x^3-5x^6+4x-7x^2+8$ is 6,
and the degree of $ax^2+abx^5+c^6x+9$ is 5, if x is the variable.

Functions of the second and third degree are also called **quadratic** and **cubic** functions respectively. Thus $ax^2+bxy+cy^2$ and $ax^3+by^3+cx^2y+dxy^2$ are respectively the quadratic and cubic functions of x, y.

Taking $a_0, a_1, a_2, a_3, \ldots\ldots$ as the co-efficients, the general expressions in x, of various degrees, are as follows:

1st degree a_0x+a_1	...	...	[2 terms.]
2nd degree $a_0x^2+a_1x+a_2$	...	...	[3 terms.]
3rd degree $a_0x^3+a_1x^2+a_2x+a_3$		...	[4 terms.]
...	...	...	...
...	...	...	...
...	...	...	...

and of the nth degree is

$$a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\ldots+a_{n-2}x^2+a_{n-1}x+a_n \ldots [n+1 \text{ terms.}]$$

Similarly, the general expression in x, y of various degrees are as follows:

1st degree $a_0x+a_1y+a_2$	...	[3 terms.]
2nd degree $a_0x^2+a_1xy+a_2y^2+a_3x+a_4y+a_5$	...	[6 terms.]
3rd degree $a_0x^3+a_1x^2y+a_2xy^2+a_3y^3+a_4x^2+a_5xy+a_6y^2+a_7x+a_8y+a_9$	...	[10 terms.]

Thus, a complete general expression of any degree should contain all the terms of the given degree, together with all the terms of lower degrees.

If we multiply a function of the 3rd degree, say $a_0x^3+a_1x^2+a_2x+a_3$ by a function of the 2nd degree, say $b_0x^2+b_1x+b_2$, the product would evidently be of the 5th degree.

Similarly, if we multiply a function of the mth degree by a function of the nth degree, the product would be of the $(m+n)^{th}$ degree.

Again, if we divide a function of the 4th degree, say $x^4+x^2y^2+y^4$, by a function of the 2nd degree, say x^2+xy+y^2, the quotient is evidently of the 2nd degree.

Similarly, if we divide a function of the mth degree by a function of the nth degree, the quotient would evidently be of the $(m-n)^{th}$ degree. These relations may be put in the form of the following law:

Law of Degree. The degree of the *product* of two integral functions is the *sum* of the degrees of the functions and the degree of the *quotient* of two integral functions is the *difference* of their degrees.

4. Method of Detached Co-efficients. When two integral functions of the *same variables* are to be multiplied together or one is to be divided by another, the process may be abbreviated, by taking the following steps, which are based upon the *law of degree*:

1. Supply the missing terms (if any), by affixing a zero to each of such terms as its co-efficient.

2. Arrange the integral functions in descending or ascending powers of the variable.

3. After removing the variable, do the process of multiplication or division as usual.

4. To the terms of the product or the quotient thus got, affix the variable with proper degree, bearing in mind the law of degree.

Example 2. Multiply x^3+3x-1 by x^2-4x+2.

The first expression when completed $=x^3+0x^2+3x-1$.

As both the expressions are already arranged in descending powers of x, we take their co-efficients in the given order.

$$\begin{array}{rrrrrrr} 1 & + & 0 & + & 3 & - & 1 \\ 1 & - & 4 & + & 2 & & \\ \hline 1 & + & 0 & + & 3 & - & 1 \\ -4 & - & 0 & -12 & + & 4 & \\ & + & 2 & + & 0 & + & 6 & - & 2 \\ \hline 1 & - & 4 & + & 5 & -13 & +10 & - & 2 \end{array}$$

As the given expressions are of the 3rd and 2nd degrees respectively, their product must be of the 5th degree.

$\therefore$ the product $=x^5-4x^4+5x^3-13x^2+10x-2$.

Example 3. Divide x^4+9x^2+81 by x^2+3x+9.

The dividend when completed $=x^4+0x^3+9x^2+0x+81$.

Taking the co-efficients in order, we have

$$\begin{array}{rrrrrrrrrr} 1+3+9) & 1 & + & 0 & + & 9 & + & 0 & +81 & (1-3+9 \\ & 1 & + & 3 & + & 9 & & & & \\ \hline & & - & 3 & + & 0 & + & 0 & & \\ & & - & 3 & - & 9 & -27 & & & \\ \hline & & & & + & 9 & +27 & +81 & & \\ & & & & + & 9 & +27 & +81 & & \\ \hline & & & & * & & * & & * & \end{array}$$

As the dividend and the divisor are of the 4th and 2nd degrees respectively, the quotient must be of the 2nd degree.

$\therefore$ quotient $=x^2-3x+9$.

The method employed in these examples is called the **method of detached co-efficients.**

EXERCISE 96.

1. Write down a general expression in x of the 8th degree.

2. Write down a general expression in x, y of the 4th degree.

Taking x, y, z as the variables, state which of the following expressions are :

(i) integral and rational,
(ii) integral and irrational,
(iii) fractional and rational,
(iv) fractional and irrational.

3. $a\sqrt{x}+\frac{b}{\sqrt{y}}+cz$.

4. $ax^2+\frac{b}{x}+\frac{1}{c}$.

5. $p\sqrt{x^3}+qx$.

6. $\left(x+\frac{1}{x}\right)\left(y+\frac{1}{y}\right)\left(z+\frac{1}{z}\right)$.

7. $\left(x+\frac{1}{a}\right)^2+\left(y+\frac{1}{b}\right)^2+\left(z+\frac{1}{c}\right)^2$.

8. $\frac{ax-b\sqrt{x^3}}{c\sqrt{x}}$.

9. If $f(x)=3x^2-4x+5$, find the value of
(i) $f(1)$, (ii) $f(2)$, (iii) $f(-1)$, (iv) $f(0)$, (v) $2f(x)$,
(vi) $3f(-k)$, (vii) $f(m+1)$.

10. If $f(x)=2x^3-x^2+3x+1$, find the value of
(i) $f(2)-f(1)$, (ii) $f(1)+f(0)$, (iii) $f(p)+f(-p)$.

11. If $f(x)=ax^2+bx+c$, find the value of $f(x+1)-f(x-1)$

12. If $f(x)=\frac{2x+1}{2x-1}$, find the value of $3f(2)-2f(0)$.

Multiply, using the method of detached co-efficients :

13. x^2-3x+2 by $2x^2+x-3$.
14. x^3-2x^2+x+3 by x^2-3x+4.
15. x^2-5x+3 by $2x^3-3x+1$.
16. $7x^4-2x^2+3$ by $2x^3-4x-3$.
17. $5x^3-2x-1$ by $3x^3+x^2+1$.
18. $5x^4-3x^3-2x^2+4x+1$ by x^3-2x^2+x-1.
19. $6x^4+x^3-1$ by $2x^4-3x^2+4$.
20. $\frac{2}{3}x^3-\frac{4}{5}x^2y+\frac{1}{2}y^3$ by $\frac{3}{5}x-2y$.
21. $\cdot4x^2+\cdot3xy-\cdot2y^2$ by $\cdot5x^2-\cdot2xy+\cdot5y^2$.
22. Expand, using detached co-efficients:
$(x^2+2x+1)(x^3-3x+1)(2x^3+x^2+3)$.
23. Multiply $(2x-3)(3x+2)(x-a)$ and find the value of a for which the co-efficient of x^2 vanishes.

Divide, using the method of detached co-efficients:

24. $x^5-2x-x^4+6x^2-4x^3$ by $2+x^3-4x$.

25. $11x^3+48-12x+30x^4-82x^2$ by $2x-4+3x^2$.

26. $1-5x+9x^2-6x^3+x^5$ by $1-2x+x^2$.

27. $8x^3-1$ by $4x^2+2x+1$.

28. x^4-1 by $x+1$.

29. Find the expression of the 4th degree which multiplied by x^3+x^2+x+1 gives x^7+x^6+x+1 as the product.

5. Homogeneous expression. An expression is said to be **homogeneous** when all its terms are of the same degree. Thus,

x^2-y^2+3xy is a homogeneous expression of the 2nd degree in x and y.

$x^2-y^2+z^2-2xy+3xz-yz$ is a homogeneous expression of the 2nd degree in x, y and z.

$ax^3+by^3+cz^3-dxyz$ is a homogeneous expression of the 3rd degree in x, y and z.

Complete Homogeneous Expression

$5x^3+4x^2y+3xy^2+y^3$ is a homogeneous expression of the 3rd degree in x and y and is **complete in itself**, for no term is missing; whereas $5x^3+3xy^2+y^3$ is also a homogeneous expression of the 3rd degree in x and y but is **incomplete**, for a term of the type x^2y is missing. It can be made complete by inserting $+0x^2y$ in it. The insertion of $+0x^2y$ does not change its value, for $0x^2y=0$.

Similarly, any incomplete expression can be put in the form as a complete expression by inserting the missing terms with zero as the co-efficient of each of such terms.

Type. Let us examine the forms of terms in a few complete homogeneous expressions in x and y:

a_0x+a_1y ... ... (i)
$a_0x^2+a_1xy+a_2y^2$ ... ... (ii)
$a_0x^3+a_1x^2y+a_2xy^2+a_3y^3$ ... ... (iii)
$a_0x^4+a_1x^3y+a_2x^2y^2+a_3xy^3+a_4y^4$ ... (iv)
$a_0x^5+a_1x^4y+a_2x^3y^2+a_3x^2y^3+a_4xy^4+a_5y^5$ ... (v)

and classify the terms in each, according to the forms or types.

In the first expression the terms are of *one* type only, *i.e.*, a_0x.

In the 2nd expression the terms are of *two* types, *i.e.*, a_0x^2 and a_1xy.

In the 3rd expression the terms are of *two* types, *i.e.*, a_0x^2 and a_1x^2y.

In the 4th expression the terms are of *three* types, *i.e.*, a_0x^4, a_1x^3y and $a_2x^2y^2$.

In the 5th expression the terms are of *three* types, *i.e.*, a_0x^5, a_1x^4y and $a_2x^3y^2$.

If in these expressions the terms of the same type be grouped together, we have

(a_0x+a_1y) ... ... (i)
$(a_0x^2+a_2y^2)+a_1xy$ ... ... (ii)
$(a_0x^3+a_3y^3)+(a_1x^2y+a_2xy^2)$ ... ... (iii)
$(a_0x^4+a_4y^4)+(a_1x^3y+a_3xy^3)+a_2x^2y^2$ ... (iv)
$(a_0x^5+a_5y^5)+(a_1x^4y+a_4xy^4)+(a_2x^3y^2+a_3x^2y^3)$... (v)

Thus, the best way of writing down a homogeneous expression is first to think out all the possible types and then to write down all the terms in each type.

Example 1. Write down a homogeneous expression of the 3rd degree in x, y and z.

The types of terms in the required expression are :

(i) ax^3, (ii) bx^2y, (iii) $cxyz$.

∴ the required expression is

$(a_0x^3+a_1y^3+a_2z^3)+(a_3x^2y+a_4x^2z+a_5y^2x+a_6y^2z+a_7z^2x+a_8z^2y)+a_9xyz$.

6. Absolute Symmetry. Just as in Geometry, a symmetrical figure is one in which all points on one side of the axis of symmetry have corresponding points on the other side, or if the points on one side be interchanged with the corresponding points on the other side, the figure remains unaltered, so in Algebra, a **symmetrical expression**, with respect to a pair of letters in it, is one which remains unaltered if those two letters be interchanged, *e.g.* $2x^2+3xy+2y^2$ is symmetrical with respect to x and y, for when x and y are interchanged, the resulting expression $2y^2+3yx+2x^2$ is equal to the original expression.

Such a symmetry is known as **absolute symmetry**.

The expression $2x^2+2y^2+2z^2+3xy+3xz+3yz$ is symmetrical with respect to (x, y), (y, z) and (z, x), for it remains unaltered when (i) x, y are interchanged, (ii) y, z are interchanged and (iii) when z, x are interchanged.

The expression $2x^2+2y^2+5z^2+3xy+7xz+7yz$ is symmetrical with respect to (x, y) but not with respect to (x, z) and (y, z), for if we interchange x and y, its value remains unaltered, but its value is altered when x and z or y and z are interchanged.

Ex. Think out the condition which will make the above expression symmetrical

(i) with respect to x, z;

(ii) with respect to y, z.

Let us consider the expressions:

$a_0x^5+a_1y^5$ ... ... (i)

$a_2x^4y+a_3xy^4$ ... ... (ii)

$a_4x^3y^2+a_5x^2y^3$ ... ... (iii)

Expression (i) will be symmetrical with respect to x, y if $a_0=a_1$.

Expression (ii) will be symmetrical with respect to x, y if $a_2=a_3$.

Expression (iii) will be symmetrical with respect to x, y if $a_4=a_5$.

Consequently, $(a_0x^5+a_1y^5)+(a_2x^4y+a_3xy^4)+(a_4x^3y^2+a_5x^2y^3)$ will be symmetrical, with respect to x, y if the terms of the type x^5 have equal co-efficients, the terms of the type x^4y have equal co-efficients and the terms of the type x^3y^2 have equal co-efficients.

Law. In a symmetrical expression all the terms of the same type must have equal co-efficients.

Let us consider the *sum*, the *difference*, the *product* and the *quotient* of symmetrical expressions.

Take the following 4 expressions, which are symmetrical with respect to x, y:

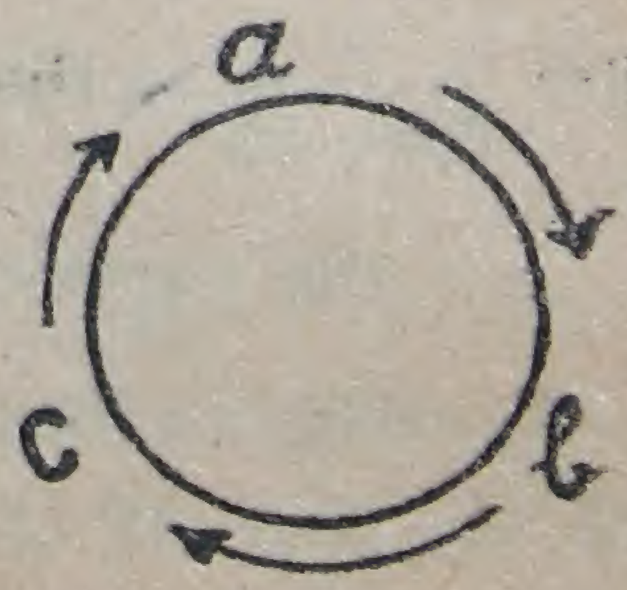
$x+y$ ... ... (i)
x^2+xy+y^2 ... ... (ii)
x^2-xy+y^2 ... ... (iii)
$x^4+x^2y^2+y^4$ ... ... (iv)

and apply the test of symmetry on the results we get by adding (i) and (ii), (i) and (iii), (i) and (iv), (ii) and (iii), (ii) and (iv), (iii) and (iv);
by subtracting (i) from (ii), (i) from (iii), (i) from (iv), (ii) from (iii), (ii) from (iv) and (iii) from (iv);
by multiplying (i) and (ii), (i) and (iii), (i) and (iv), (ii) and (iii), (ii) and (iv), (iii) and (iv);
and by dividing (iv) by (ii) and (iv) by (iii).

From the examination of the above results, we establish the following law:

Law. The sum, the difference, the product and the quotient of two symmetrical expressions are symmetrical.

7. Cyclic Order. In this figure, the letters a, b, c, are so arranged round the circumference of a circle that if we start from a and go round the circumference in the direction of the arrows, we meet with b and after b with c and after c with a, and so on. Such an arrangement of letters is said to be **cyclic.**



Let us consider the arrangement of terms in the following expressions:

$$a(b+c)+b(c+a)+c(a+b),$$
$$a^2(b-c)+b^2(c-a)+c^2(a-b).$$

In these expressions, if in the first terms a be replaced by b, b by c and c by a, we get the second terms; if in the second terms b be replaced by c, c by a and a by b, we get the third terms; if in the third terms c be replaced by a, a by b and b by c, we get the first terms. Such expressions are said to be **written in cyclic order.**

An expression is said to be written in **cyclic order** when each term in it can be derived from the preceding term by changing the first letter into the second, the second into the third and the third into the first.

Example 2. Write $a(c^2-b^2)+b(a^2-c^2)+c(b^2-a^2)$ in cyclic order.

The first factors of the terms, *i.e.*, a, b, c are already in cyclic order, but the second factors are not.

Arranging the second factors of the terms in cyclic order, we get $-a(b^2-c^2)-b(c^2-a^2)-c(a^2-b^2)$

$$=-[a(b^2-c^2)+b(c^2-a^2)+c(a^2-b^2)].$$

Example 3. Write down the cyclic expression in a, b, c whose first term is $\dfrac{a}{a(b-c)}$.

Putting b for a, c for b and a for c, we get $\dfrac{b}{b(c-a)}$ as the second term.

Putting, in the second term, c for b, a for c and b for a, we get $\dfrac{c}{c(a-b)}$ as the third term.

$$\therefore \quad \text{the expression} = \frac{a}{a(b-c)}+\frac{b}{b(c-a)}+\frac{c}{c(a-b)}.$$

8. Sigma Notation. Whenever an expression consists of the sum of terms of the same type, it is quite easy to

derive all the terms from the first term, as illustrated in example 3. Such an expression can be put in an abbreviated form by prefixing the Greek letter Σ (Sigma) to the first term. The symbol Σ stands for *the sum of the terms of the same type.*

Thus, with variables a, b, c,

Σa means $a+b+c$,

Σa^2 means $a^2+b^2+c^2$,

Σab means $ab+bc+ca$,

$\Sigma(a^2+ab)$ means $(a^2+ab)+(b^2+bc)+(c^2+ca)$,

$3\Sigma ab$ means $3(ab+bc+ca)$,

and $2\Sigma a^2-3\Sigma ab$ means $2(a^2+b^2+c^2)-3(ab+bc+ca)$.

9. Cyclic Symmetry. Let us examine the nature of the expression $a^2(b-c)+b^2(c-a)+c^2(a-b)$. At first sight it looks as if it were absolutely symmetrical with respect to the letters (a, b), (b, c) and (c, a) but on interchanging a, b, we get the expression

$$b^2(a-c)+a^2(c-b)+c^2(b-a)$$

which $$=-a^2(b-c)-b^2(c-a)-c^2(a-b).$$

Thus, this expression differs from the original expression, in having each of its terms preceded by a negative sign instead of a positive sign ; hence the original expression *does not possess absolute symmetry*. However, it possesses a certain kind of symmetry which is known as the **cyclic symmetry**. Even such a symmetry is of great importance, as it helps us in deriving unknown results from one which is known, as illustrated in the next example.

Example 4. Simplify by the cyclic symmetry

$$(a+b-2c)(a+c-b)+(b+c-2a)(b+a-c)+(c+a-2b)(c+b-a).$$

By actual multiplication,

$$(a+b-2c)(a+c-b)=a^2-b^2-2c^2-ac+3bc.$$

By cyclic symmetry,

$$(b+c-2a)(b+a-c)=b^2-c^2-2a^2-ba+3ca$$

and $(c+a-2b)(c+b-a)=c^2-a^2-2b^2-cb+3ab)$

$\therefore$ the expression $=-2(a^2+b^2+c^2)+2(ab+ac+bc)$

$$=-2\Sigma a^2+2\Sigma ab.$$

EXERCISE 97.

Taking x, y, z as variables state which of the functions given below are

(i) homogeneous but not symmetrical,
(ii) symmetrical but not homogeneous,
(iii) homogeneous and symmetrical,
(iv) neither homogeneous nor symmetrical.

In cases (ii) and (iii), tell also whether the symmetry is absolute or cyclic:

1. $(x+y)^3$.
2. $(x-y)^3$.
3. $(x-y)(x+y-6)$.
4. $x^3-2x^2y+2xy^2+y^3$.
5. $x^3+ax^2y+bxy^2+y^3$.
6. $(x^2+y^2)(x^3-y^3)$.
7. $x^3+y^3+z^3-3xyz$.
8. $2(x^2+y^2)+5xy+a(x+y)$.
9. $x^3+3x^2y+3xy^2-y^3$.
10. $x^2+y^2+z^2-a(xy+yz+zx)$.
11. $x^2+y^2+z^2-2(x+y+z)$.
12. $xy(x-y)+yz(y-z)+zx(z-x)$.
13. Write down all the possible *types* of terms in a complete homogeneous expression of the 5th degree in x and y.
14. Write down all the possible types of terms in a complete homogeneous expression of the 4th degree in x, y and z.
15. Write down a homogeneous and symmetrical expression of the 4th degree in x, y and z, using a, b, c, &c. as co-efficients.
16. Construct a homogeneous and symmetrical expression of the 2nd degree in x and y, which shall be equal to 16 when $x=1$ and $y=2$, and which shall be equal to 4 when $x=2$ and $y=-1$.
17. Construct a homogeneous and symmetrical expression of the 3rd degree in x and y, which shall be equal to 10 when $x=1$ and $y=1$, and equal to 36 when $x=2$ and $y=1$.

Write down all the types of terms in the following expressions, a, b, c being the variables and x being constant:

18. $\Sigma(x+a)(x+b)$.
19. $\Sigma(x-a)(b-a)(c-a)$.
20. $\Sigma(x+a)(x+b)(x-c)$.
21. $\Sigma\dfrac{x^2(a+b)}{ab}$.

22. $\Sigma \frac{bc}{(a-b)(a-c)}$.

23. $\Sigma \frac{1}{(a-b)(a-c)(x-a)}$.

Simplify the following expressions by the method of cyclic symmetry :

24. $(x+b)(x+c)+(x+c)(x+a)+(x+a)(x+b)$.

25. $(a+b-c)(b-c)+(b+c-a)(c-a)+(c+a-b)(a-b)$.

26. $(x+b+c)(x+b-c)+(x+c+a)(x+c-a)$
$+(x+a+b)(x+a-b)$.

27. $(a-b)(x+a)(x+b)+(b-c)(x+b)(x+c)$
$+(c-a)(x+c)(x+a)$.

28. $\Sigma(x+y)(x+z)$.

29. $\Sigma(x+y)(x-z)$.

30. $\Sigma xy(x-y)(x-z)$.

31. $\Sigma(a+b-c)^2$.

Under what conditions will the following expressions be symmetrical :

32. (i) $7x^2+ay^2+bz^2-3xy+cyz+dzx$,

(ii) $4x^3+py^3-2x^2y+qxy^2$,

(iii) $2x^4+my^4-nz^4+7x^2yz-py^2zx+qz^2xy$.

Without actual multiplication, state why the following relations are wrong :

33. $(a+b)(a^2-ab+b^2)=a^3-b^3$.

34. $a^3+b^3=(a-b)(a^2+ab+b^2)$.

35. $(a+b+c)(a^2+b^2+c^2-ab-bc-ac)$
$=(a^3+b^3+c^3-2ab-2bc-2ca)$.

36. $(x^2+y^2)(x+y)=x^3+2xy^2+3x^2y+y^3$.

37. $(x^3+y^3)(x+y)=x^4+x^3y+3xy^3+y^4$.

38. $(x+y+z)(x^2+y^2+z^2)=(x^3+y^3+z^3+xy+xz+zy)$.

39. $(x+y)(x^2+xy+y^2)=x^3+3xy+3xy^2+y^4$.

40. $(x+y+1)(x+y+z)=x^2+xy+xz+y^2$.

10. Principle of Indeterminate Co-efficients.

Let us compare the nature of relation in the following examples

$x+3=8$ ... ... (i)

$2x-1=7$ ... ... (ii)

$$(2x-4)+(3x-2)=5x-6 \quad \ldots \quad \ldots \quad \ldots \quad \text{(iii)}$$

$$(x+1)(x-1)=x^2-1 \quad \ldots \quad \ldots \quad \ldots \quad \text{(iv)}$$

$$\frac{2x^2+3x+7}{x+1}=(2x+1)+6 \quad \ldots \quad \ldots \quad \text{(v)}$$

(i) is true for $x=5$ and for no other value of x,
(ii) is true for $x=4$ and for no other value of x,
(iii) is true for *any value* of x, say 0, 1, 2, 3, 4, 5, 6, &c.
(iv) is true for *any value* of x, say 0, 1, 2, 3, 4, 5, 6, &c.
(v) is true for *any value* of x, say 0, 1, 2, 3, 4, 5, 6, &c.

(i) and (ii) are equations, for they are true for definite values of x.

(iii), (iv) and (v) are identities, for they are true for any value of x.

Def. An identity is true for *any* value of the variable or variables.

Symbol $\equiv$ is used for '*identically equal to*'.

As the idea of identity implies that the two expressions *differ only in form*, it follows that the *co-efficient of any power of a variable in one expression is equal to the co-efficient of the same power of the variable in the other expression.* This is known as the **principle of indeterminate co-efficients** but really it should be called the **principle of undetermined co-efficients.**

Although the principle stated above is quite obvious, a formal proof is added below:

Proof. Let $a_0+a_1x+a_2x^2+a_3x^3+\ldots\equiv A_0+A_1x+A_2x^2+A_3x^3+\ldots$ be an identity and hence true for any value of x.

Put $x=0$, then $a_0=A_0$.

Since $a_0=A_0$

$\therefore \quad a_1x+a_2x^2+a_3x^3+\ldots\equiv A_1x+A_2x^2+A_3x^3\ldots$

Dividing both sides by x, we get

$a_1+a_2x+a_3x^2+\ldots\equiv A_1+A_2x+A_3x^2+\ldots$

Put $x=0$, then $a_1=A_1$.

Similarly, $a_2=A_2$, $a_3=A_3$, &c.

Hence the *co-efficients of like powers of x are equal.*

The principle established here is of great importance.

Example 1. If $a_0x^3+a_1x^2+a_2x+a_3 \equiv (x^2-3x+2)(x+3)$, find the values of a_0, a_1, a_2, a_3.

$$a_0x^3+a_1x^2+a_2x+a_3 \equiv (x^2-3x+2)(x+3)$$
$$\equiv x^3-7x+6.$$

Equating the co-efficients of terms of the same degree, we get $a_0=1$, $a_1=0$, $a_2=-7$, $a_3=6$.

The co-efficient of the missing term is taken to be 0.

Example 2. If $2x^2+9x+6 \equiv Ax(x+1)+Bx(x+2)+C(x+1)(x+2)$, find the values of A, B and C.

$$2x^2+9x+6 \equiv A(x^2+x)+B(x^2+2x)+C(x^2+3x+2)$$
$$\equiv (A+B+C)x^2+(A+2B+3C)x+2C.$$

Equating the co-efficients of terms of equal powers, we get

$A+B+C=2$ ... ... (i)

$A+2B+3C=9$ ... ... (ii)

$2C=6$ ... ... (iii)

From (iii), $C=3$.

Substituting the value of C in (i) and (ii), we get

$A+B+3=2$

or $A+B=-1$ ... ... (iv)

and $A+2B+9=9$

or $A+2B=0$ ... ... (v)

From (iv) and (v), $A=-2$ and $B=1$.

$\therefore$ $A=-2$, $B=1$ and $C=3$.

Alternative Method

Instead of equating the co-efficients of terms of equal powers, it is sometimes more convenient to find the values of the constants, by assigning suitable values to x, as illustrated below:

In $2x^2+9x+6 \equiv Ax(x+1)+Bx(x+2)+C(x+1)(x+2)$,

put $x=0$; then $2\times0+9\times0+6=A\times0+B\times0+C\times1\times2$

or $6=2C$

$\therefore$ $C=3$.

Again, by putting -1 for x in the above identity, we get

$$2(-1)^2+9(-1)+6=A(-1)(0)+B(-1)(-1+2)$$
$$+C(0)(-1+2)$$
$$\text{or} \quad 2-9+6=-B$$
$$\therefore \quad B=1$$

Again, by putting -2 for x in the above identity, we get

$$2(-2)^2+9(-2)+6=A(-2)(-2+1)+B(-2)(0)+$$
$$C(-2+1)(0)$$
$$\text{or} \quad 8-18+6=2A$$
$$\therefore \quad A=-2.$$
$$\therefore \quad A=-2, \quad B=1 \text{ and } C=3.$$

Sometimes the first method is convenient and sometimes the second. It is only by practice that we can find out which method is to be employed in a particular case.

Example 3. Simplify $(a+b)^3+(b+c)^3+(c+a)^3$ $-3(a+b)(b+c)(c+a)$ by the principles of homogeneity, symmetry and indeterminate co-efficients.

Since the given expression is symmetrical, homogeneous and of the 3rd degree in a, b, c, the result will also satisfy these conditions.

The types of terms in the result will be

(i) a^3, (ii) a^2b and (iii) abc.

$$\therefore \quad (a+b)^3+(b+c)^3+(c+a)^3-3(a+b)(b+c)(c+a)$$
$$\equiv p(a^3+b^3+c^3)+q(a^2b+a^2c+b^2c+b^2a+c^2a+c^2b)$$

$+r.abc$, where p, q, r are the co-efficients.

Putting $a=0, b=0$ and $c=1$, we get

$$(0)^3+(1)^3+(1)^3-3(0)(1)(1)\equiv p(1)+q(0)+r(0)$$
$$\therefore \quad p=2.$$

Putting $a=0, b=1$ and $c=1$, we get

$$(1)^3+(2)^3+(1)^3-3(1)(2)(1)\equiv p(2)+q(1+1)+r(0)$$
$$10-6=2p+2q$$
$$=2\times 2+2q$$
$$q=0.$$

Again, putting $a=2$, $b=1$ and $c=1$, we get

$$(3)^3+(2)^3+(3)^3-3(3)(2)(3)\equiv p(8+1+1)+q(4+4+1+2+2+1)+r.\ 2.\ 1.\ 1.$$

or $27+8+27-54\equiv p\times 10+q\times 14+r\times 2.$

Substituting the values of p and q, we get

$$8\equiv 2\times 10+0\times 14+2r$$

or $r=-6.$

$\therefore$ $p=2$, $q=0$ and $r=-6.$

Hence the given expression $=2(a^3+b^3+c^3)-6abc.$

EXERCISE 98.

1. If $ax^2+bx+c\equiv 2(x+3)(x+1)-9$; find the values of a, b and c.

2. If $ax^3+bx^2+cx+d\equiv 3(x+1)(x-2)^2+3$, find the values of a, b, c and d.

3. If $lx^3+mx^2+nx+k\equiv(x+2)(x+3)(x-4)+3x+1$, find the values of l, m, n and k.

4. If $ax^4+bx^3+cx^2+dx+e\equiv(x+1)^4+(x-1)^4$, find the values of a, b, c, d and e.

5. If $a_0x^4+a_1x^3+a_2x^2+a_3x+a_4\equiv 2(x-2)(x+1)^3-10$, find the values of a_0, a_1, a_2, a_3 and a_4.

6. For what value of m will $x+3$ be a factor of x^2+9x+m?

7. If $x^3-6x^2+12x+k\equiv(x-a)^3$, find the values of k and a.

Find the numerical values of A, B, C and D in the following identities:

8. $9x-7\equiv A(x-1)+B(4x-3).$

9. $12x+5\equiv A(3x+4)-B(2x-1).$

10. $2(3x^2-13x+13)\equiv A(x-1)(x-2)+B(x-1)(x-3)+C(x-2)(x-3).$

11. $3x^2+11x-7\equiv(Ax+B)(x-2)+C(x^2+3x-1).$

12. $2x^3-x^2+x-2\equiv A(x^2-1)(x+2)+B(x-1)(x^2-4)+C(x+1)(x^2-4)+D(x^2-1)(x-2).$

13. Express $3x^2-4x+8$ in the form of
$$A(x-1)^2+B(x+2)+C.$$
14. Express $3x^3+9x^2+7x+2$ in the form of
$$A(x+1)^3+B(x+1)+C.$$
15. Express x^3-x^2-4x-6 in the form of $A(x+1)^3$
$$+B(x+1)^2+C(x+1)+D.$$
16. If $(x^2+y^2+z^2)(x+y+z)\equiv A\Sigma x^3+B\Sigma x^2y+C\Sigma xyz$, find the values of A, B and C.

Simplify :

17. $(x+y+z)^3-x^3-y^3-z^3$.
18. $(x+y+z)^3-(y+z-x)^3-(z+x-y)^3-(x+y-z)^3$.
19. $(y+z-2x)(y-z)^2+(z+x-2y)(z-x)^2$
$$+(x+y-2z)(x-y)^2+(y+z-2x)(z+x-2y)(x+y-2z).$$

Example 4. Find by the method of indeterminate co-efficients the square root of
$$16x^6-8x^5+x^4-16x^3+4x^2+4.$$

The square root must evidently be of the 3rd degree and its first term is $4x^3$.

Hence the form of the square root is $4x^3+ax^2+bx+c$.

Now, we have to find the values of a, b, c, so as to satisfy the identity $16x^6-8x^5+x^4-16x^3+4x^2+4\equiv(4x^3+ax^2+bx+c)^2$.

Using the method of detached co-efficients and avoiding the unnecessary terms, we get the corresponding co-efficients

$$\begin{array}{r} 4+\ a+\ b+\ c \\ 4+\ a+\ b+\ c \\ \hline 16+4a+4b+4c \\ \quad +4a+a^2+ab+\ \ldots \\ \quad\quad +4b+ab+\ \ldots \\ \quad\quad\quad +4c+ \\ \hline 16+8a+(8b+a^2)+(2ab+8c)+\ \ldots \end{array}$$

Equating the co-efficients of like powers, we have

(i) $8a=-8$ $\quad\therefore\ a=-1.$
(ii) $8b+a^2=1$ $\quad$ or $8b+(-1)^2=1,$ $\quad\therefore\ b=0.$
(iii) $2ab+8c=-16$ $\quad$ or $2(-1)(0)+8c=-16,$ $\quad\therefore\ c=-2.$

Substituting the values of a, b, c in $4x^3+ax^2+bx+c$, we get the required root $=4x^3-x^2-2$.

Find the square root of the following expressions by the method of indeterminate co-efficients ;

20. $9x^4-6x^3+13x^2-4x+4$.

21. $9x^6-12x^5+4x^4+24x^3-16x^2+16$.

22. $4x^6-12x^4+20x^3+9x^2-30x+25$.

23. Find the values of a and b for which $x^4+2ax^3-3x^2+2bx+4$ is a perfect square.

24. Find the values of p and q for which $9x^4+px^3+qx^2-10x+1$ is a perfect square.

Example 5. Find the cube root of $x^6+3x^5+9x^4+13x^3+18x^2+12x+8$ by the method of indeterminate co-efficients.

Since the expression is of the 6th degree, its cube root must evidently be of the 2nd degree and its first term $=\sqrt[3]{x^6}$ or x^2.

Hence the form of the cube root is x^2+ax+b.

Now, we have to find out the values of a and b, so as to satisfy the identity

$$x^6+3x^5+9x^4+13x^3+18x^2+12x+8\equiv(x^2+ax+b)^3.$$

Using the method of detached co-efficients and avoiding the unnecessary terms, we get the corresponding co-efficients.

$$\begin{array}{l}
1+\ a+b\\
1+\ a+b\\
\hline
1+\ a+b\\
\quad+\ a+a^2+\ldots\\
\qquad\quad +b\ +\ldots\\
\hline
1+2a+(a^2+2b)+\ldots\\
1+\ a+b\\
\hline
1+2a+(a^2+2b)\ +\ldots\\
\quad+\ a+\qquad 2a^2+\ldots\\
\qquad\quad +\qquad\ \ b\ +\ldots\\
\hline
1+3a+(3a^2+3b)+\ldots
\end{array}$$

Equating the co-efficients of like powers, we have

(i) $3a=3$, $\quad\therefore\quad a=1$.

(ii) $3a^2+3b=9$, or $3.1^2+3b=9$, $\quad\therefore\quad b=2$.

Substituting the values of a and b in x^2+ax+b,

we get the required cube root $=x^2+x+2$.

Find the cube root of the following expressions by the method of indeterminate co-efficients :

25. $a^6-3a^5+9a^4-13a^3+18a^2-12a+8$.

26. $x^6-9x^5+21x^4+9x^3-42x^2-36x-8$.

27. $8x^6+12x^5-30x^4-35x^3+45x^2+27x-27$.

28. $27-54x+63x^2-44x^3+21x^4-6x^5+x^6$.

29. Find p so that $x^6+px^5+5x^3+px-1$ may be a perfect cube.

30. For what values of a and b will the expression $8x^6+12x^5-30x^4-5ax^3+5(a+2b)x^2+9(a-4b)x-27$ be a perfect cube ?

Example 6. Express $\frac{x+4}{(x+1)(x+2)}$ in the form of $\frac{A}{x+1}+\frac{B}{x+2}$.

Putting $\frac{x+4}{(x+1)(x+2)}\equiv\frac{A}{x+1}+\frac{B}{x+2}$, we get

$$x+4\equiv A(x+2)+B(x+1)$$
$$\equiv Ax+2A+Bx+B$$
$$\equiv x(A+B)+(2A+B).$$

Comparing the co-efficients, we have

$A+B=1$ ... ... (i)

and $2A+B=4$ ... ... (ii)

From (i) and (ii), $A=3$ and $B=-2$.

Hence the required form is $=\frac{3}{x+1}-\frac{2}{x+2}$.

31. Express $\frac{2(1-x)}{(1+x)(x^2-x-6)}$ in the form of $\frac{A}{x+1}+\frac{B}{x-3}+\frac{C}{x+2}$.

32. Express $\frac{2(x+5)}{8x^3+1}$ in the form of $\frac{A}{2x+1}+\frac{Bx+C}{4x^2-2x+1}$.

33. Express $\frac{9}{(x-2)(x+1)^2}$ in the form of $\frac{A}{x-2}+\frac{Bx+C}{(x+1)^2}$.

34. If $\frac{2x^2-x-3}{(x-1)^2(x-2)}\equiv\frac{A}{x-2}+\frac{B}{x-1}+\frac{C}{(x-1)^2}$, find the values of A, B and C.

35. If $\frac{x^2+x+2}{(x^2+x+1)(x+1)^2} \equiv \frac{A}{(x+1)} + \frac{B}{(x+1)^2} + \frac{Cx+D}{x^2+x+1}$, find the values of A, B, C and D.

11. We now illustrate the combined application of the factor theorem, law of degree, homogeneity, symmetry and the principle of indeterminate co-efficients.

Example 1. Factorise x^3-5x^2-5x-6.

As the expression is of the 3rd degree, it can have either three linear factors or one linear and one quadratic, *i.e., it must have at least one linear factor.* [Law of degree.]

As the first term of the expression is x^3.

$\therefore$ the first term of the linear factor is x.

As the last term of the expression is -6.

$\therefore$ the last term of the linear factor may be one of

$$\pm 1, \pm 2, \pm 3, \pm 6.$$

Substituting $\mp 1, \mp 2, \mp 3$, for x in the given expression, we find that it does not vanish, but on substituting $+6$ for x, we see that it vanishes.

$\therefore$ $x-6$ is one of the factors. [Factor theorem.]

The given expression can be put as

$$x^2(x-6)+x(x-6)+(x-6)$$
$$=(x-6)(x^2+x+1).$$

As x^2+x+1 cannot be further factorised,

$\therefore$ $(x-6)$, (x^2+x+1) are the required factors.

Example 2. Factorise x^3+x^2+4x-6.

As the expression is of the 3rd degree, it must have at least one linear factor.

As the first term of the expression is x^3, therefore the first term of the linear factor is x.

As the last term of the expression is -6, therefore the last term of the linear factor may be one of $\pm 1, \pm 2, \pm 3, \pm 6$.

On trial we find that the given expression vanishes when we substitute $+1$ for x,

$\therefore$ $(x-1)$ is one of the factors.

Now, the expression $= x^2(x-1)+2x(x-1)+6(x-1)$

$=(x-1)(x^2+2x+6)$.

As x^2+2x+6 cannot be further factorised,

$\therefore$ $(x-1), (x^2+2x+6)$ are the required factors.

Example 3. Factorise x^3-5x^2+2x+8.

As the expression is of the third degree, with x^3 as its first term and $+8$ as its last term, therefore the first term of the linear factor is x and the last term of the linear factor may be one of $\pm 1, \pm 2, \pm 4, \pm 8$.

On trial we find that the expression vanishes when $x=-1, +2$ and $+4$,

$\therefore$ $(x+1)$, $(x-2)$, $(x-4)$ are the three factors of the expression.

As these three linear factors form a cubic expression and the given expression is also cubic, therefore there will be no other factor.

EXERCISE 99.

Factorise:

1. x^3+x^2+x-3.
2. $x^3-6x^2+11x-6$.
3. $x^3+x^2-17x+15$.
4. $x^3+3x^2-10x-24$.
5. $x^3+10x^2+31x+30$.
6. x^3+5x^2-x-5.
7. $x^3-19x-30$.
8. x^3-x^2+12.
9. $x^4-2x^3-3x^2+2x+2$.
10. $x^4-2x^3-20x^2+39x+36$.
11. If n denote any odd number, prove that $(b+c)(c+a)(a+b)$ is a factor of $(a+b+c)^n-a^n-b^n-c^n$ and that $(b-c)(c-a)(a-b)$ is a factor of $(b-c)^n+(c-a)^n+(a-b)^n$.

12. If n denote any odd number which is not a multiple of 3, prove that $ab(a+b)(a^2+ab+b^2)$ is a factor of $(a+b)^n - a^n - b^n$.

Example 4. Factorise $(x+y+z)^3 - x^3 - y^3 - z^3$.

As the given expression is symmetrical and of the 3rd degree in x, y, z, the possible forms of the linear factor are

(i) $(x+y+z)$, (ii) $(x-y)$, (iii) $(x+y)$, (iv) x.

Putting $-(y+z)$ for x in the given expression, we find that it does not vanish,

$\therefore$ $x+y+z$ is not a factor of the expression.

Putting y for x in the given expression, we find that it does not vanish,

$\therefore$ $x-y$ is not a factor of the expression.

Putting $-y$ for x in the given expression, we find that it vanishes,

$\therefore$ $x+y$ is a factor of the expression.

As the given expression is symmetrical and homogeneous, therefore $(y+z)$ and $(z+x)$ must *necessarily* be two other factors.

As the given expression is of the 3rd degree, it cannot have any other factor excepting a numerical one.

$\therefore (x+y+z)^3 - x^3 - y^3 - z^3 \equiv A(x+y)(y+z)(z+x)\ldots$ (i)

where A stands for the numerical factor.

If in (i) $x=0$, $y=1$ and $z=1$, then

$$(0+1+1)^3 - 0^3 - 1^3 - 1^3 = A(0+1)(1+1)(1+0)$$

$$A = 3.$$

$\therefore$ $(x+y+z)^3 - x^3 - y^3 - z^3 \equiv 3(x+y)(y+z)(z+x)$.

Example 5. Factorise $(xy+yz+zx)^2 - x^2y^2 - y^2z^2 - z^2x^2$.

As the given expression is symmetrical and of the fourth degree in x, y, z, the possible forms of the linear factors, if any, are

(i) x, (ii) $x-y$, (iii) $x+y$, (iv) $x+y+z$.

Putting 0 for x, we find that the expression vanishes,

$\therefore$ x is one of the factors.

As the expression is symmetrical and homogeneous,
$\therefore$ y and z must be two other factors.

As the given expression is of the 4th degree, symmetrical and homogeneous, and the expression xyz formed by the factors x, y, z is of the 3rd degree, symmetrical and homogeneous, the remaining factor must be of the 1st degree, symmetrical and homogeneous, which cannot be other than $A(x+y+z)$ where A is the numerical co-efficient.

$\therefore (xy+yz+zx)^2-x^2y^2-y^2z^2-z^2x^2 \equiv Axyz(x+y+z)$...(i)
Putting in (i) $x=1, y=1, z=1$, we get $3A=6$ or $A=2$.
$\therefore$ $(xy+yz+zx)^2-x^2y^2-y^2z^2-z^2x^2=2xyz(x+y+z)$.

Example 6. Factorise $a^3(b^2-c^2)+b^3(c^2-a^2)+c^3(a^2-b^2)$.

As the given expression is symmetrical and of the 5th degree in a, b, c, the possible forms of the linear factors, if any, are
(i) a, (ii) $a-b$, (iii) $a+b$, (iv) $a+b+c$.
Putting 0 for a, we find that the expression does not vanish,
$\therefore$ a is not a factor of the expression.
Putting b for a, we find that the expression vanishes,
$\therefore$ $(a-b)$ is one of the factors.
As the expression is symmetrical and homogeneous,
$\therefore$ $(b-c)$ and $(c-a)$ must also be factors.

As the expression is of the 5th degree, symmetrical and homogeneous, and the expression $(a-b)(b-c)(c-a)$ formed by the factors is of the 3rd degree, there will be one more factor of the 2nd degree, in a, b, c which will be symmetrical, homogeneous and of the form $[A(a^2+b^2+c^2)+B(ab+bc+ca)]$.

$\therefore$ $a^3(b^2-c^2)+b^3(c^2-a^2)+c^3(a^2-b^2)$
$\equiv(a-b)(b-c)(c-a)\{A(a^2+b^2+c^2)+B(ab+bc+ca)\}$.

Comparing like terms, say a^3b^2, on both sides, we get
$a^3b^2=-a^2b\times Bab=-Ba^3b^2$
$\therefore$ $B=-1$.

Again, comparing terms containing a^4b, we get

$$0 = -a^2b \times Aa^2 = -Aa^4b$$

$$\therefore \quad A = 0.$$

$$\therefore \quad a^3(a^2-b^2)+b^3(c^2-b^2)+c^3(a^2-b^2)$$

$$=(a-b)(b-c)(c-a)\{(0)(a^2+b^2+c^2)+(-1)(ab+bc+ca)\}$$

$$=-(a-b)(b-c)(c-a)(ab+bc+ca).$$

In the process of this example, the evaluation of the constants, by giving different values to the variables is tedious, therefore we have adopted the method of the *comparison of like terms.*

NOTE. In finding out the value of the constants, care should be taken not to give such values to the variables as to reduce the two sides to the form $0=0$.

13. Prove that $a^2(b-c)+b^2(c-a)+c^2(a-b) \equiv -(a-b)(b-c)(c-a)$.

14. Prove that $ab(a-b)+bc(b-c)+ca(c-a) \equiv -(a-b)(b-c)(c-a)$.

[Learn these two formulæ by heart.]

Factorise the following expressions using questions 11, 12, wherever necessary :

15. $a(b^2-c^2)+b(c^2-a^2)+c(a^2-b^2)$.
16. $(a-b)^3+(b-c)^3+(c-a)^3$.
17. $(a+b+c)(ab+ac+bc)-abc$.
18. $a^2(b+c)+b^2(c+a)+c^2(a+b)+2abc$.
19. $(a+b+c)^3-(b+c-a)^3-(c+a-b)^3-(a+b-c)^3$.
20. $a^3(b-c)+b^3(c-a)+c^3(a-b)$.
21. $a(b-c)^3+b(c-b)^3+c(a-b)^3$.
22. $a(b+c)^2+b(c+a)^2+c(a+b)^2-3abc$.
23. $(a+b)(a-b)^3+(b+c)(b-c)^3+(c+a)(c-a)^3$.
24. $(x-y)^5+(y-z)^5+(z-x)^5$.
25. $a^4(b-c)+b^4(c-a)+c^4(a-b)$.
26. $(x+y)^5-x^5-y^5$.
27. $(x+y+z)^5-x^5-y^5-z^5$.
28. $x^4(y^2-z^2)+y^4(z^2-x^2)+z^4(x^2-y^2)$.
29. $(x+y+z)^4-(x+y)^4-(y+z)^4-(z+x)^4+x^4+y^4+z^4$.

CHAPTER XXIII

MISCELLANEOUS ARTIFICES

In this chapter it is proposed to discuss

(i) a few important conditional identities with their application in factors and fractions,

(ii) harder types of factors, and

(iii) the application of the remainder theorem and the method of cross-multiplication in H.C.F. and L.C.M.

1. Conditional identities based on $a+b+c=0$.

Example 1. If $a+b+c=0$, shew that $(a+b)(b+c)(c+a) = -abc$.

Since $a+b+c=0$, $\therefore$ $a+b=-c$, $b+c=-a$, $c+a=-b$.

Multiplying these three results, we get

$$(a+b)(b+c)(c+a)=(-c)(-a)(-b)$$
$$=-abc.$$

Example 2. If $a+b+c=0$, shew that

$$a^2-bc=b^2-ca=c^2-ab.$$

Since $a+b+c=0$, $\therefore$ $a=-b-c$

$$a^2-bc=a\times a-bc$$
$$=a(-b-c)-bc$$
$$=-ab-ac-bc=-(ab+bc+ca).$$

Similarly, $b^2-ca=-(ab+bc+ca)$,

and $c^2-ab=-(ab+bc+ca)$.

$\therefore$ $a^2-bc=b^2-ca=c^2-ab$.

Example 3. If $a+b+c=0$, shew that $a^3+b^3+c^3=3abc$.

Since $a+b+c=0$, $\therefore$ $a+b=-c$

$\therefore$ $(a+b)^3=(-c)^3$

$\therefore$ $a^3+b^3+3ab(a+b)=-c^3$

or $a^3+b^3+c^3=-3ab(a+b)$

$$=-3ab(-c)=3abc.$$

Example 4. If $a+b+c=0$, shew that

$$a^5+b^5+c^5=-5abc(ab+ac+bc).$$

Since $a+b+c=0$, $\therefore$ $a+b=-c$.

$\therefore \quad (a+b)^5=(-c)^5$

$\therefore \quad a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5=-c^5$

$a^5+b^5+c^5=-5ab(a^3+2a^2b+2ab^2+b^3)$

$=-5ab\{(a+b)^3-ab(a+b)\}$

$=-5ab\{(-c)^3-ab(-c)\}$

$=-5ab(-c^3+abc)$

$=-5abc(ab-c^2)$

$=-5abc(ab-c\times c)$

$=-5abc\{ab+c(a+b)\}$

$=-5abc(ab+ac+bc).$

EXERCISE 100.

If $a+b+c=0$, shew that

1. $ab(a+b)=bc(b+c)=ca(c+a).$
2. $a^2+b^2+c^2=-2(ab+ac+bc).$
3. $a(b^2+c^2-a^2)=b(c^2+a^2-b^2)=c(a^2+b^2-c^2)=-2abc.$
4. $a^2+ab+b^2=b^2+bc+c^2=c^2+ca+a^2.$
5. $(a^4+b^4+c^4)=\frac{1}{2}(a^2+b^2+c^2)^2.$
6. $a^4+b^4+c^4=2(ab+bc+ca)^2.$
7. $\frac{a^5+b^5+c^5}{5}=\frac{a^3+b^3+c^3}{3}\cdot\frac{a^2+b^2+c^2}{2}.$
8. $a^5+b^5+c^5=5abc(c^2-ab).$

Example 5. If $2s=a+b+c$, then $s^2+(s-a)^2+(s-b)^2+(s-c)^2=a^2+b^2+c^2.$

The left-hand side $=s^2+s^2-2sa+a^2+s^2-2sb+b^2+s^2-2sc+c^2$

$=4s^2-2s(a+b+c)+a^2+b^2+c^2$

$=4s^2-2s.2s+a^2+b^2+c^2$

$=4s^2-4s^2+a^2+b^2+c^2$

$=a^2+b^2+c^2.$

If $2s=a+b+c$, shew that

9. $(s-a)^3+(s-b)^3+(s-c)^3+3abc=s^3.$
10. $s(s-b)(s-c)+s(s-c)(s-a)+s(s-a)(s-b)$
$=(s-a)(s-b)(s-c)+abc.$
11. $2(s-a)(s-b)(s-c)+a(s-b)(s-c)+b(s-c)(s-a)$
$+c(s-a)(s-b)=abc.$
12. $(s-a)^3+(s-b)^3+(s-c)^3-3(s-a)(s-b)(s-c)$
$=\frac{1}{2}(a^3+b^3+c^3-3abc).$

HARDER FACTORS

2. Type XIII. Factors of cyclic expressions by arrangement of terms.

Example 1. Factorise $a^3(b-c)+b^3(c-a)+c^3(a-b)$.

Arranging the terms, according to descending powers of a, we have

$$\begin{aligned}\text{the expression} &= a^3(b-c)-a(b^3-c^3)+(b^3c-c^3b)\\ &= a^3(b-c)-a(b^2+bc+c^2)(b-c)+bc(b+c)(b-c)\\ &= (b-c)\{a^3-ab^2-abc-ac^2+b^2c+bc^2\}.\end{aligned}$$

Arranging the 2nd factor, according to descending powers of b, we have

$$\begin{aligned}\text{the expression} &= (b-c)\{(b^2c-ab^2)+(bc^2-abc)-(ac^2-a^3)\}\\ &= (b-c)\{b^2(c-a)+bc(c-a)-a(c+a)(c-a)\}\\ &= (b-c)(c-a)\{b^2+bc-ac-a^2\}.\end{aligned}$$

Arranging the 3rd factor, according to descending powers of c, we have

$$\begin{aligned}\text{the expression} &= (b-c)(c-a)\{(bc-ac)+(b^2-a^2)\}\\ &= (b-c)(c-a)\{c(b-a)+(b+a)(b-a)\}\\ &= (b-c)(c-a)(b-a)(a+b+c)\\ &= -(a-b)(b-c)(c-a)(a+b+c).\end{aligned}$$

EXERCISE 101.

Resolve into factors :

1. $a^2(b-c)+b^2(c-a)+c^2(a-b)$.
2. $a^2(b+c)+b^2(c+a)+c^2(a+b)+2abc$.
3. $a^4(b-c)+b^4(c-a)+c^4(a-b)$.
4. $a^3(b^2-c^2)+b^3(c^2-a^2)+c^3(a^2-b^2)$.
5. $a^4(b^2-c^2)+b^4(c^2-a^2)+c^4(a^2-b^2)$.
6. $a^5(b-c)+b^5(c-a)+c^5(a-b)$.

Type XIV. Factors of expressions of the form ax^2+bx+c which involve the application of the equations

$$p+q=m$$
$$pq=n^2.$$

Example. 2 Factorise $20x^2+xy-30y^2$.

Here we have to find out two numbers, p and q such that

$$p+q=+1 \quad \ldots \quad \ldots \quad \text{(i)}$$

and

$$pq=20\times(-30)=-600 \quad \ldots \quad \text{(ii)}$$

$$\therefore \quad (p-q)^2=(p+q)^2-4pq$$

$$=1^2-4(-600)$$

$$=2401$$

$$\therefore \quad p-q=\sqrt{2401}=49 \quad \ldots \quad \text{(iii)}$$

Adding (i) and (iii), and subtracting (iii) form (i) we get

$$p=25 \text{ and } q=-24.$$

$$\therefore \quad 20x^2+xy-30y^2=20x^2+25xy-24xy-30y^2$$

$$=5x(4x+5y)-6y(4x+5y)$$

$$=(4x+5y)\,(5x-6y).$$

Factorise :

7. $18x^2-51xy+35y^2$.
8. $27a^2-33ab-20b^2$.
9. $55x^2-13xy-12y^2$.
10. $120x^4y^2+73x^3y^3-28x^2y^4$.
11. $44x^2-71xy+20y^2$.
12. $21x^2-58xy+21y^2$.
13. $28x^2-73xy-20y^2$.

Type XV. Factors of expressions of the form ax^2+bx+c, with literal co-efficients.

Example 3. Factorise $a^2-(x+y)a+(x+1)(y-1)$.

Here we have to find out two quantities whose product $=(x+1)(y-1)$, and whose sum $=-(x+y)$.

Obviously, $-(x+1)$ and $-(y-1)$ are the required quantities.

$$\therefore \quad a^2-(x+y)a+(x+1)(y-1)$$

$$=a^2-(x+1)a-(y-1)a+(x+1)(y-1)$$

$$=a\{a-(x+1)\}-(y-1)\{a-(x+1)\}$$

$$=(a-x-1)\,(a-y+1).$$

Resolve into factors :

14. $a^2+a(x+y)+(x-2)(2+y)$.
15. $(a-1)x^2-x-a$.
16. $x^2+x-(a+1)(a+2)$.
17. $x^2-(1+a+b)x+a(1+b)$.
18. $(b^2-1)x^2+(b^2+1)x+b$.
19. $a^2-(2b+1)a+b^2+b-6$.
20. $x^2+2px+p^2-1$.
21. $x^2+(a+b+c)x+ab+ac$.

Type XVI. Factors of cubic expressions with one term missing.

Example 4. Factorise $x^3-19x-30$.

$$\begin{aligned} x^3-19x-30 &= x^3-19x-38+8 \\ &= (x)^3+(2)^3-19(x+2) \\ &= (x+2)(x^2-2x+4)-19(x+2) \\ &= (x+2)(x^2-2x+4-19) \\ &= (x+2)(x^2-2x-15) \\ &= (x+2)(x-5)(x+3). \end{aligned}$$

Example 5. Factorise $2x^3-3x^2-4$.

$$\begin{aligned} 2x^3-3x^2-4 &= 2x^3-3x^2-16+12 \\ &= (2x^3-16)-3(x^2-4) \\ &= 2(x^3-8)-3(x+2)(x-2) \\ &= 2(x^2+2x+4)(x-2)-3(x+2)(x-2) \\ &= (x-2)\{2x^2+4x+8-3x-6\} \\ &= (x-2)(2x^2+x+2). \end{aligned}$$

Resolve into factors:

22. $x^3-13x+12$.
23. x^3-6x^2+32.
24. $x^3-3x-18$.
25. $x^3-5x+12$.
26. $x^3-3x^2y+20y^3$.
27. $8x^3+4x-3$.
28. $2a^3-a^2b-b^3$.
29. $27x^3+12x-5$.

Type XVII. Factors of expressions of the form $ax^4+bx^3+cx^2+bx+a$ or $ax^4+bx^3y+cx^2y^2+bxy^3+ay^4$.

Example 6. Factorise $8x^4-6x^3+7x^2-6x+8$.

$$\text{The expression} = x^2\left(8x^2-6x+7-\frac{6}{x}+\frac{8}{x^2}\right)$$

$$=x^2\left[8\left(x^2+\frac{1}{x^2}\right)-6\left(x+\frac{1}{x}\right)+7\right]$$

$$=x^2\left[8\left(x^2+2+\frac{1}{x^2}\right)-6\left(x+\frac{1}{x}\right)+7-16\right]$$

$$=x^2\left[8\left(x+\frac{1}{x}\right)^2-6\left(x+\frac{1}{x}\right)-9\right]$$

Putting A **for** $\left(x+\frac{1}{x}\right)$, the expression

$$=x^2[8A^2-6A-9]$$
$$=x^2[8A^2-12A+6A-9]$$
$$=x^2[4A(2A-3)+3(2A-3)]$$
$$=x^2(2A-3)(4A+3).$$

Substituting the value of A, we have

$$\text{the expression}=x^2\left[2\left(x+\frac{1}{x}\right)-3\right]\left[4\left(x+\frac{1}{x}\right)+3\right]$$
$$=x^2\left[\frac{2x^2-3x+2}{x}\right]\left[\frac{4x^2+3x+4}{x}\right]$$
$$=(2x^2-3x+2)(4x^2+3x+4).$$

NOTE. It may be noted that in such expressions, the co-efficients of terms equi-distant from the middle term are equal, irrespective of signs, and their special characteristic is that when they are equated to zero, the equations thus formed have reciprocal roots: hence such expressions are known as **Reciprocal Expressions.**

Resolve into factors :

30. $x^4-x^3-8x^2+x+1$.
31. $x^4+4x^3-10x^2+4x+1$.
32. $x^4-5x^3+6x^2-5x+1$.
33. $2x^4+3x^3+5x^2+3x+2$.
34. $3x^4-x^3y-8x^2y^2+xy^3+3y^4$.
35. $4x^4+8x^3y+3x^2y^2+8xy^3+4y^4$.
36. $6x^4-25x^3y+7x^2y^2+25xy^3+6y^4$.

Type XVIII. Factors of homogeneous expressions of the second degree involving three letters.

Example 7. Factorise $2a^2+3ab-9ac-2b^2+7bc-5c^2$.

If $c=0$, the expression is reduced to

$$2a^2+3ab-2b^2=(2a-b)(a+2b) \quad \text{... (i)}$$

If $b=0$, the expression is reduced to

$$2a^2-9ac-5c^2=(2a+c)(a-5c) \quad \text{... (ii)}$$

If $a=0$, the expression is reduced to

$$-2b^2+7bc-5c^2=(-2b+5c)(b-c)$$
$$=(2b-5c)(-b+c) \quad \text{... (iii)}$$

Factors in (i), (ii) and (iii) are incomplete and complementary. Their proper association can be determined by comparing the co-efficients and signs of like terms.

Here combining (i) and (ii) and comparing the results with (iii), we have

$$2a^2+3ab-9ac-2b^2+7bc-5c^2$$
$$=(2a-b+c)(a+2b-5c).$$

Resolve into factors :

37. $2a^2-ab-6b^2+ac+19bc-15c^2$.
38. $2x^2-2y^2-3z^2+3xy-5xz+5yz$.
39. $x^2-xy+4zx+yz-2y^2+3z^2$.
40. $a^2+(x+y)a-2x^2+5xy-2y^2$.
41. $x(x+a)-2a^2+3b(a+x)+2b^2$.
42. $x^2-xy+4xz-2y^2-5yz+3z^2$.
43. $2x^2+5xy-3xz+5yz-3y^2-2z^2$.

Type XIX. Factors of expressions which satisfy the condition $a+b+c=0$.

Example 8. Factorise $(x-y)^3+(y-z)^3+(z-x)^3$.

Since $(x-y)+(y-z)+(z-x)=0$,

$\therefore\ (x-y)^3+(y-z)^3+(z-x)^3=3(x-y)(y-z)(z-x)$.

[Example 3. Conditional identities. p. 436.]

Example 9. Factorise $(x-2y)^5+(2y-3z)^5+(3z-x)^5$.

Since $(x-2y)+(2y-3z)+(3z-x)=0$,

$\therefore \quad (x-2y)^5+(2y-3z)^5+(3z-x)^5$

$=-5(x-2y)(2y-3z)(3z-x)\ \{\ (x-2y)(2y-3z)+(x-2y)(3z-x)+(2y-3z)(3z-x)\}$.

[Example 4. Conditional identities, p. 436.]

Resolve into factors:

44. $(2a-b)^3+(b-3c)^3+(3c-2a)^3$.

45. $(a-2b+1)^3+(a+b-1)^3+(b-2a)^3$.

46. $(a-1)^3+(a-2)^3-(2a-3)^3$.

47. $(a+b+1)^3+(a-b-1)^3-8a^3$.

48. $(2x+y)^3-(x-y)^3-(x+2y)^3$.

49. $a^3(b-c)^3+b^3(c-a)^3+c^3(a-b)^3$.

50. $(x-y)^5+(y-z)^5+(z-x)^5$.

51. $(x-1)^5+(x-2)^5-(2x-3)^5$.

Type XX. Miscellaneous.

Resolve into factors:

52. $a^2(b+c)+b^2(c+a)+c^2(a+b)+3abc$.

[*Hint.* $3abc=abc+abc+abc$.

Exp. $=[a^2(b+c)+abc]+$two similar terms.]

53. $a(b^2+c^2-a^2)+b(c^2+a^2-b^2)+c(a^2+b^2-c^2)+6abc$.

54. $a^2(b+c)+b^2(c+a)+c^2(a+b)+a^3+b^3+c^3$.

55. $2(a^2b^2+a^2c^2+b^2c^2)-a^4-b^4-c^4$.

[*Hint.* Exp. $=4a^2b^2-(a^4+b^4+c^4+2a^2b^2-2b^2c^2-2c^2a^2)$

$=(2ab)^2-(a^2+b^2-c^2)^2$.]

56. $(a+b)(a+c)(b+c)+abc$.

[*Hint.* Exp. $=[a^2+a(b+c)+bc](b+c)+abc$

$=a^2(b+c)+a(b+c)^2+bc(b+c)+abc$

$=[a^2(b+c)+abc]+[a(b+c)^2+bc(b+c).]$

57. $x^4-5x^3+14x^2-20x+16$.

[*Hint.* Exp. $=(x^4+8x^2+16)-5x(x^2+4)+6x^2$.]

58. $x^4+8x^3+24x^2+32x-20$.

3. Fractions with denominators in cyclic order.

It is useful to learn by heart the following formulæ, after proper verification :

If a, b, c are the variables

(i) $\Sigma a = a+b+c$.

(ii) $\Sigma(a+b) = 2(a+b+c)$.

(iii) $\Sigma(a-b) = 0$.

(iv) $\Sigma(a^2-b^2) = 0$.

(v) $\Sigma a(b-c) = 0$.

(vi) $\Sigma a^2(b-c) = -(a-b)(b-c)(c-a)$.

(vii) $\Sigma ab(b-c) = -(a-b)(b-c)(c-a)$.

(viii) $\Sigma a(b^2-c^2) = (a-b)(b-c)(c-a)$.

(ix) $\Sigma(a-b)^3 = 3(a-b)(b-c)(c-a)$.

(x) $\Sigma a^3(b-c) = -(a-b)(b-c)(c-a)(a+b+c)$.

(xi) $\Sigma a^4(b-c) = -(a-b)(b-c)(c-a)(a^2+b^2+c^2+ab+bc+ca)$.

(xii) $\Sigma a^3(b^2-c^2) = -(a-b)(b-c)(c-a)(ab+bc+ca)$.

Example 1.

Simplify $\dfrac{1}{(a-b)(a-c)} + \dfrac{1}{(b-c)(b-a)} + \dfrac{1}{(c-a)(c-b)}$.

Putting the denominators in cyclic order, we have the expressions

$$= -\frac{1}{(a-b)(c-a)} - \frac{1}{(b-c)(a-b)} - \frac{1}{(c-a)(b-c)}$$

$$= -\left\{\frac{1}{(a-b)(c-a)} + \frac{1}{(b-c)(a-b)} + \frac{1}{(c-a)(b-c)}\right\}$$

$$= -\frac{(b-c)+(c-a)+(a-b)}{(a-b)(b-c)(c-a)}$$

$$= -\frac{0}{(a-b)(b-c)(c-a)}$$

$$= 0.$$

Example 2.

Simplify $\dfrac{a^3}{(a-b)(a-c)} + \dfrac{b^3}{(b-a)(b-c)} + \dfrac{c^3}{(c-a)(c-b)}$.

Putting the denominators in cyclic order, we have the expression

$$= -\frac{a^3}{(a-b)(c-a)} - \frac{b^3}{(a-b)(b-c)} - \frac{c^3}{(c-a)(b-c)}$$

$$= -\left\{\frac{a^3}{(a-b)(c-a)} + \frac{b^3}{(a-b)(b-c)} + \frac{c^3}{(c-a)(b-c)}\right\}$$

$$= -\frac{a^3(b-c)+b^3(c-a)+c^3(a-b)}{(a-b)(b-c)(c-a)}$$

$$= -\frac{-(a-b)(b-c)(c-a)(a+b+c)}{(a-b)(b-c)(c-a)}$$

$$= +\frac{(a-b)(b-c)(c-a)(a+b+c)}{(a-b)(b-c)(c-a)}$$

$$= a+b+c.$$

EXERCISE 102.

Simplify :

1. $\dfrac{a}{(a-b)(a-c)} + \dfrac{b}{(b-c)(b-a)} + \dfrac{c}{(c-a)(c-b)}$.

2. $\dfrac{a^2}{(a-b)(a-c)}$ + two similar terms.

3. $\dfrac{bc}{(a-b)(a-c)}$ + two similar terms.

4. $\dfrac{1}{a(a-b)(a-c)} + \dfrac{1}{b(b-c)(b-a)} + \dfrac{1}{c(c-a)(c-b)}$.

5. $\dfrac{1}{bc(a-b)(a-c)} + \dfrac{1}{ca(b-c)(b-a)} + \dfrac{1}{ab(c-a)(c-b)}$.

6. $\Sigma \dfrac{a}{bc(a-b)(a-c)}$.

7. $\dfrac{x-a}{(a-b)(a-c)} + \dfrac{x-b}{(b-c)(b-a)} + \dfrac{x-c}{(c-a)(c-b)}$.

8. $\Sigma \dfrac{x^2-yz}{(x-y)(x-z)}$.

9. $\Sigma \dfrac{x^2+yz}{(x-y)(x-z)}$.

10. $\Sigma \dfrac{b^2c^2}{(a-b)(a-c)}$.

11. $\Sigma \dfrac{a^4}{(a-b)(a-c)}$.

12. $\Sigma \dfrac{bc(b+c)}{(a-b)(a-c)}$.

13. $\Sigma \dfrac{b^2+c^2}{(a^2-b^2)(a^2-c^2)}$.

14. $\Sigma \dfrac{bc}{a(a^2-b^2)(a^2-c^2)}$.

15. $\Sigma \dfrac{a(a+b)(a+c)}{(a-b)(a-c)}$.

16. $\Sigma \dfrac{(a+1)^2}{(a-b)(a-c)}$.

17. $\dfrac{b-c}{b+c}+\dfrac{c-a}{c+a}+\dfrac{a-b}{a+b}$.

18. $\dfrac{(x-b)(x-c)}{(a-b)\,(a-c)}$ + two similar terms.

19. $\dfrac{(a+b)(b+c)(c+a)}{abc}-\dfrac{a+b}{c}-\dfrac{b+c}{a}-\dfrac{c+a}{b}$.

20. $\dfrac{b^2+c^2-2a^2}{(a-b)(a-c)}+\dfrac{c^2+a^2-2b^2}{(b-c)\,(b-a)}+\dfrac{a^2+b^2-2c^2}{(c-a)(c-b)}$.

[*Hint.* $b^2+c^2-2a^2=(b^2-a^2)-(a^2-c^2)$. Decompose and factorise.]

21. $\dfrac{bc(x-a)^2}{(a-b)(a-c)}+\dfrac{ca(x-b)^2}{(b-c)(b-a)}+\dfrac{ab(x-c)^2}{(c-a)(c-b)}$.

Example 3. Simplify $\dfrac{a}{(a-b)(a-c)(x-a)}+\dfrac{b}{(b-a)(b-c)(x-b)}+\dfrac{c}{(c-a)(c-b)(x-c)}$.

Putting the denominators in cyclic order, we have

the expression $=-\dfrac{a}{(a-b)(c-a)(x-a)}-\dfrac{b}{(a-b)(b-c)(x-b)}-\dfrac{c}{(c-a)(b-c)(x-c)}$

$\therefore$

$$=-\left\{\frac{a}{(a-b)(c-a)(x-a)}+\frac{b}{(a-b)(b-c)(x-b)}+\frac{c}{(c-a)(b-c)(x-c)}\right\}$$

$$=-\frac{a(b-c)(x-b)(x-c)+\text{two similar terms}}{(a-b)(b-c)(c-a)(x-a)(x-b)(x-c)}.$$

The numerator $=a(b-c)\{x^2-x(b+c)+bc\}+$two similar terms

$$=x^2\Sigma a(b-c)-x\Sigma a(b^2-c^2)+abc\Sigma(b-c)$$
$$x^2\times 0-x\times(a-b)(b-c)(c-a)+abc\times 0$$
$$=-x(a-b)(b-c)(c-a).$$

$\therefore$ the expression $=-\dfrac{-x(a-b)(b-c)(c-a)}{(a-b)(b-c)(c-a)(x-a)(x-b)(x-c)}$

$$=\frac{x}{(x-a)(x-b)(x-c)}.$$

Simplify :

22. $\dfrac{1}{(a-b)(a-c)(x-a)}+\dfrac{1}{(b-c)(b-a)(x-b)}+\dfrac{1}{(c-a)(c-b)(x-c)}$.

23. $\dfrac{a^2}{(a-b)(a-c)(x-a)}+\dfrac{b^2}{(b-c)(b-a)(x-b)}+\dfrac{c^2}{(c-a)(c-b)(x-c)}$.

24. $\dfrac{1}{(a-b)(a-c)(x+a)}+\dfrac{1}{(b-c)(b-a)(x+b)}+\dfrac{1}{(c-a)(c-b)(x+c)}$

25. $\dfrac{a^2}{(a-b)(a-c)(x+a)}+\dfrac{b^2}{(b-c)(b-a)(x+b)}+\dfrac{c^2}{(c-a)(c-b)(x+c)}$.

26. $\dfrac{1+a}{(a-b)(a-c)(x-a)}+\dfrac{1+b}{(b-c)(b-a)(x-b)}+\dfrac{1+c}{(c-a)(c-b)(x-b)}$.

[*Hint.* Decompose the expression into $\dfrac{1}{(a-b)(a-c)(x-a)}+$ two similar terms $+\dfrac{a}{(a-b)(a-c)(x-a)}+$ two similar terms.]

27. $\dfrac{1+pa+qa^2}{(a-b)(a-c)(x-a)}+\dfrac{1+pb+qb^2}{(b-c)(b-a)(x-b)}+\dfrac{1+pc+qc^2}{(c-a)(c-b)(x-c)}$.

Example 4. Simplify $\dfrac{(a-b)^3+(b-c)^3+(c-a)^3}{a^3(b-c)^3+b^3(c-a)^3+c^3(a-b)^3}$.

The expression $=\dfrac{(a-b)^3+(b-c)^3+(c-a)^3}{(ab-ac)^3+(bc-ba)^3+(ca-cb)^3}$.

Since $(a-b)+(b-c)+(c-a)=0$ and $(ab-ac)+(bc-ba)+(ca-cb)=0$,

$$\therefore \text{ the expression} = \frac{3(a-b)(b-c)(c-a)}{3(ab-ac)(bc-ba)(ca-cb)}$$

$$= \frac{(a-b)(b-c)(c-a)}{abc(a-b)(b-c)(c-a)}$$

$$= \frac{1}{abc}.$$

Simplify :

28. $\dfrac{(a^2-b^2)^3+(b^2-c^2)^3+(c^2-a^2)^3}{(a-b)^3+(b-c)^3+(c-a)^3}$.

29. $\dfrac{a^3(b^2-c^2)+b^3(c^2-a^2)+c^3(a^2-b^2)}{a^2(b-c)+b^2(c-a)+c^2(a-b)}$.

30. $\dfrac{27(a+b+c)^3-(a+2b)^3-(b+2c)^3-(c+2a)^3}{(a+3b+2c)(b+3c+2a)(c+3a+2b)}$.

[*Hint.* Apply $(a+b+c)^3-a^3-b^3-c^3=3(a+b)(b+c)(c+a)$.]

31. $\dfrac{(a+b)^3+(c+d)^3-(a+c)^3-(b+d)^3}{(a+b)^2+(c+d)^2-(a+c)^2-(b+d)^2}$.

32. $\dfrac{(a-b)(a+b)^3+(b-c)(b+c)^3+(c-a)(c+a)^3}{(a+b)(a-b)^3+(b+c)(b-c)^3+(c+a)(c-a)^3}$.

33. $\dfrac{x^2\left(\frac{1}{y}-\frac{1}{z}\right)+y^2\left(\frac{1}{z}-\frac{1}{x}\right)+z^2\left(\frac{1}{x}-\frac{1}{y}\right)}{x\left(\frac{1}{y}-\frac{1}{z}\right)+y\left(\frac{1}{z}-\frac{1}{x}\right)+z\left(\frac{1}{x}-\frac{1}{y}\right)}$.

34. Shew that $\dfrac{a-b}{1+ab}+\dfrac{b-c}{1+bc}+\dfrac{c-a}{1+ca}$

$$= \frac{a-b}{1+ab}\times\frac{b-c}{1+bc}\times\frac{c-a}{1+ca}.$$

35. Shew that $\dfrac{a-b}{m+ab}+\dfrac{b-c}{m+bc}+\dfrac{c-a}{m+ca}$

$$= m.\frac{a-b}{m+ab}\cdot\frac{b-c}{m+bc}\cdot\frac{c-a}{m+ca}.$$

36. Prove that $\dfrac{1}{(a-b)^2}+\dfrac{1}{(b-c)^2}+\dfrac{1}{(c-a)^2}$

$$= \left\{\frac{1}{a-b}+\frac{1}{b-c}+\frac{1}{c-a}\right\}^2.$$

4. Conditional Evaluation.

Example 1. Reduce to the simplest terms.

$$\frac{x+y}{1-xy} \text{ and } \frac{x-y}{1+xy}$$

where $x = \frac{a+b}{1-ab}$ and $y = \frac{a-b}{1+ab}$.

$$x+y = \frac{a+b}{1-ab} + \frac{a-b}{1+ab} = \frac{(a+b)(1+ab)+(a-b)(1-ab)}{1-a^2b^2}$$

$$= \frac{2a(1+b^2)}{1-a^2b^2} \quad \ldots \quad \ldots \quad \text{(i)}$$

$$1-xy = 1 - \frac{a+b}{1-ab} \times \frac{a-b}{1+ab} = 1 - \frac{a^2-b^2}{1-a^2b^2}$$

$$= \frac{1-a^2b^2-a^2+b^2}{1-a^2b^2}$$

$$= \frac{(1-a^2)(1+b^2)}{1-a^2b^2} \quad \ldots \quad \ldots \quad \text{(ii)}$$

$$\therefore \quad \frac{x+y}{1-xy} = \frac{2a(1+b^2)}{1-a^2b^2} \div \frac{(1-a^2)(1+b^2)}{1-a^2b^2}$$

$$= \frac{2a(1+b^2)}{1-a^2b^2} \times \frac{1-a^2b^2}{(1-a^2)(1+b^2)}$$

$$= \frac{2a}{1-a^2}.$$

Similarly, $\frac{x-y}{1+xy} = \frac{2b}{1-b^2}$.

Example 2. Find the value of $\frac{x+a}{x-a} + \frac{x+b}{x-b}$ when $x = \frac{2ab}{a+b}$.

$$\text{The expression} = \frac{x-a+2a}{x-a} + \frac{x-b+2b}{x-b}$$

$$= 1 + \frac{2a}{x-a} + 1 + \frac{2b}{x-b}$$

$$= 2 + 2\left\{\frac{a}{x-a} + \frac{b}{x-b}\right\}$$

$$= 2 + 2.\frac{ax-ab+bx-ba}{(x-a)(x-b)}$$

$$= 2 + 2.\frac{x(a+b)-2ab}{(x-a)(x-b)}.$$

$$= 2 + 2.\frac{0}{(x-a)(x-b)}$$

$\because$ $x(a+b)=2ab$, or $x(a+b)-2ab=0$.

$\therefore$ the expression $=2$.

EXERCISE 103.

Find the value of:

1. $\dfrac{1}{x-a}+\dfrac{1}{x-b}-\dfrac{1}{a}$ when $x=\dfrac{2ab}{a+b}$.

2. $\dfrac{a}{2am-2mx}+\dfrac{b}{2mb-2mx}$ when $x=\dfrac{a+b}{2}$.

3. $\dfrac{x^2-y^2+x}{y^2-x^2+y}$ when $x=\dfrac{a-b}{a+b}$ and $y=\dfrac{a+b}{a-b}$.

4. $\left(\dfrac{2x-a}{2x-b}\right)^2-\dfrac{a-x}{b-x}$ when $x=\dfrac{ab}{a+b}$.

5. $\left(\dfrac{x}{x+1}\right)^2+\left(\dfrac{x}{x-1}\right)^2$ when $x^2=\dfrac{a^2}{a^2-1}$.

6. $\dfrac{x+2a}{x-2a}+\dfrac{x+2b}{x-2b}$ when $x=\dfrac{4ab}{a+b}$.

7. $\dfrac{x+2a}{2b-x}+\dfrac{x-2a}{2b+x}+\dfrac{4ab}{x^2-4b^2}$ when $x=\dfrac{ab}{a+b}$.

5. CONDITIONAL IDENTITIES.

Example 1. If $a+b+c=0$, shew that

$$\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right)\left(\frac{a+b}{c}+\frac{b+c}{a}+\frac{c+a}{b}\right)=9.$$

Since $a+b+c=0$. $\therefore$ $\left.\begin{aligned} a+b&=-c\\ b+c&=-a\\ c+a&=-b. \end{aligned}\right\}$

Substituting these values in the left-hand side, we get

$$\left\{\frac{a}{-a}+\frac{b}{-b}+\frac{c}{-c}\right\}\left\{\frac{-c}{c}+\frac{-a}{a}+\frac{-b}{b}\right\}$$

$=(-1-1-1)(-1-1-1)=9.$

Example 2. If $a+b+c=0$, shew that

$$\frac{1}{b^2+c^2-a^2}+\frac{1}{c^2+a^2-b^2}+\frac{1}{a^2+b^2-c^2}=0.$$

Since $a+b+c=0$, $\therefore \quad a+b=-c$

$\therefore \quad a^2+2ab+b^2=c^2$

$\therefore \quad a^2+b^2-c^2=-2ab.$

Similarly, $b^2+c^2-a^2=-2bc,$

and $c^2+a^2-b^2=-2ac,$

Hence the expression $= \dfrac{1}{-2bc}+\dfrac{1}{-2ac}+\dfrac{1}{-2ab}$

$= \dfrac{-a-b-c}{2abc} = \dfrac{-(a+b+c)}{2abc}$

$= \dfrac{0}{2abc}=0.$

Example 3. If $2s=a+b+c$, shew that

$$\frac{1}{s-a}+\frac{1}{s-b}+\frac{1}{s-c}-\frac{1}{s}=\frac{abc}{s(s-a)(s-b)(s-c)}.$$

The left-hand side

$$=\left(\frac{1}{s-a}+\frac{1}{s-b}\right)+\left(\frac{1}{s-c}-\frac{1}{s}\right)$$

$$=\frac{(s-b)+(s-a)}{(s-a)(s-b)}+\frac{s-s+c}{s(s-c)}$$

$$=\frac{2s-a-b}{(s-a)(s-b)}+\frac{c}{s(s-c)}$$

$$=\frac{c}{(s-a)(s-b)}+\frac{c}{s(s-c)} \qquad [\because\ 2s-a-b=c.]$$

$$=c\left\{\frac{1}{(s-a)(s-b)}+\frac{1}{s(s-c)}\right\}$$

$$=c\left\{\frac{s(s-c)+(s-a)(s-b)}{s(s-a)(s-b)(s-c)}\right\}$$

$$=c\left\{\frac{s^2-sc+s^2-sa-sb+ab}{s(s-a)(s-b)(s-c)}\right\}$$

$$=c\left[\frac{2s^2-s(a+b+c)+ab}{s(s-a)(s-b)(s-c)}\right]$$

$$=c\ \frac{2s^2-2s^2+ab}{s(s-a)(s-b)(s-c)} \qquad [\because\ a+b+c=2s.]$$

$$=\frac{abc}{s(s-a)(s-b)(s-c)}.$$

Example 4. If $x+\frac{1}{y}=1$, $y+\frac{1}{z}=1$, prove that $z+\frac{1}{x}=1$ and $xyz+1=0$.

Since $y+\frac{1}{z}=1$, $\therefore\ y=1-\frac{1}{z}=\frac{z-1}{z}$.

Substituting the value of y in $x+\frac{1}{y}=1$, we get

$$x+\frac{z}{z-1}=1$$

$$\therefore \quad \frac{zx-x+z}{z-1}=1$$

or $$xz-x+z=z-1$$

or $$xz+1=x.$$

Dividing both sides by x, we get

$$z+\frac{1}{x}=1 \quad \ldots \quad \ldots \quad \ldots \quad \text{(i)}$$

From (i) $z=1-\frac{1}{x}$

$$\therefore \quad xyz=xy\left(1-\frac{1}{x}\right).$$

$$=xy-y$$

$$=y(x-1)$$

$$=y\times\left(-\frac{1}{y}\right) \qquad \left[\because\ x-1=-\frac{1}{y}\right]$$

$$=-1$$

$$\therefore \quad xyz+1=0. \qquad \text{(ii)}$$

Example 5. If $xy+yz+zx=1$, shew that

$$\frac{x+y}{1-xy}+\frac{y+z}{1-yz}+\frac{z+x}{1-zx}=\frac{1}{xyz}.$$

Since $xy+yz+zx=1$, $\therefore\ 1-xy=z(x+y)$.

$$\therefore \quad \frac{x+y}{1-xy}=\frac{x+y}{z(x+y)}=\frac{1}{z} \quad \ldots \quad \ldots \quad \ldots \quad \text{(i)}$$

Similarly, $$\frac{y+z}{1-yz}=\frac{1}{x} \quad \ldots \quad \ldots \quad \ldots \quad \text{(ii)}$$

and $$\frac{z+x}{1-zx}=\frac{1}{y} \quad \ldots \quad \ldots \quad \ldots \quad \text{(iii)}$$

∴ Adding (i), (ii) and (iii), we get

the expression $= \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$

$$= \frac{yz+xz+xy}{xyz}$$

$$= \frac{1}{xyz}. \qquad [\because yz+xz+xy=1.]$$

EXERCISE 104.

If $a+b+c=0$, shew that :

1. $\frac{a^2+b^2-c^2}{2ab} + \frac{b^2+c^2-a^2}{2bc} + \frac{c^2+a^2-b^2}{2ca} = -3.$

2. $\frac{a^2}{2a^2+bc} + \frac{b^2}{2b^2+ca} + \frac{c^2}{2c^2+ab} = 1.$

[*Hint.* $2a^2+bc = a^2+a(-b-c)+bc$. Factorise it.]

If $2s=a+b+c$, shew that :

3. $\frac{a}{s-a} + \frac{b}{s-b} + \frac{c}{s-c} + 2 = \frac{abc}{(s-a)(s-b)(s-c)}.$

4. $1-\left[\frac{b^2+c^2-a^2}{2bc}\right]^2 = \frac{4s(s-a)(s-b)(s-c)}{b^2c^2}.$

5. If $a=1-\frac{1}{b}$, $b=1-\frac{1}{c}$, prove that $c=1-\frac{1}{a}$.

6. If $x=b+c-a$, $y=c+a-b$ and $z=a+b-c$,
then $\frac{x^3+y^3+z^3-3xyz}{a^3+b^3+c^3-3abc} = 4.$

7. If $x=a(b-c)$, $y=b(c-a)$, $z=c(a-b)$,
shew that $\left(\frac{x}{a}\right)^3 + \left(\frac{y}{b}\right)^3 + \left(\frac{z}{c}\right)^3 = \frac{3xyz}{abc}.$

8. If $\frac{y+x}{y-x} + \frac{y+z}{y-z} = 2$, then $\frac{1}{x} + \frac{1}{z} = \frac{2}{y}.$

9. If $x+y=2z$, then $\frac{x}{x-z} + \frac{y}{y-z} = 2.$

10. If $b^2=ac$, $x=\frac{1}{2}(a+b)$, $y=\frac{1}{2}(b+c)$, then $\frac{a}{x} + \frac{c}{y} = 2.$

11. If $3(a^2+b^2+c^2)=(a+b+c)^2$, then $a=b=c$.

12. If $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\frac{1}{a+b+c}$, then $\frac{1}{a^5}+\frac{1}{b^5}+\frac{1}{c^5}=\frac{1}{(a+b+c)^5}$.

13. If $x=by+cz$, $y=cz+ax$ and $z=ax+by$,

then $$\frac{a}{1+a}+\frac{b}{1+b}+\frac{c}{1+c}=1.$$

6. H. C. F. and L. C. M. (*Continued.*)

We add here a few examples and exercises which involve the application of the *remainder theorem*, *the method of cross-multiplication*, and the *principles underlying the processes of H. C. F. and L. C. M.*

Example 1. For what value of x will the expressions $4x^3-8x^2+x+3$ and $4x^3-4x^2-5x+3$ vanish?

The H. C. F. of the given expressions, when worked out $=2x-3$. As both the expressions have $2x-3$ as their common factor, they will vanish when $2x-3=0$.

or $$x=\tfrac{3}{2}.$$

Example 2. The H. C. F. and the L. C. M. of two integral expressions, each of the second degree, are $x+3$ and x^3-7x+6 respectively. Find the expressions.

Let A and B be the two expressions and Q_1 and Q_2 be respectively the quotients, when they are divided by the H. C. F. $x+3$.

$\therefore$ $A=(x+3)(Q_1)$ and $B=(x+3)(Q_2)$.

Their L. C. M. $=(x+3)(Q_1)(Q_2)$.

But as the given L. C. M. $=x^3-7x+6$,

$\therefore$ $(x+3)(Q_1)(Q_2)=x^3-7x+6$

$\therefore$ $$Q_1Q_2=\frac{x^3-7x+6}{x+3}=x^2-3x+2$$

$$=(x-1)(x-2).$$

Q_1, Q_2 could either be 1 and x^2-3x+2

or $(x-1)$ and $(x-2)$.

If we assume the first pair of values, we get expressions of the first and third degrees, which is contrary to the data.

$\therefore$ Q_1 and Q_2 are equal to $(x-1)$ and $(x-2)$.

$\therefore$ the expressions are (i) $(x+3)(x-1)=x^2+2x-3$

(ii) $(x+3)(x-2)=x^2+x-6$.

Example 3. Shew that if ax^2+bx+c and $a'x^2+b'x+c'$ have a common factor of the form $x+p$, then

$$(ca'-c'a)^2=(bc'-b'c)(ab'-a'b).$$

As $x+p$ is a factor of ax^2+bx+c as well as $a'x^2+b'x+c'$,

$\therefore$ each will vanish if $x=-p$

$$ap^2-bp+c=0 \quad \ldots \quad \ldots \quad \text{(i)}$$

$\therefore$

$$a'p^2-b'p+c'=0 \quad \ldots \quad \ldots \quad \text{(ii)}$$

By the method of cross-multiplication, from (i) and (ii), we get

$$\frac{p^2}{-bc'+b'c}=\frac{p}{ca'-c'a}=\frac{1}{-ab'+a'b}$$

$$\therefore \quad \left(\frac{p}{ca'-c'a}\right)^2=\frac{p^2}{-bc'+b'c}\times\frac{1}{-ab'+a'b}$$

$$\therefore \quad \frac{p^2}{(ca'-c'a)^2}=\frac{p^2}{+(bc'-b'c)(ab'-a'b)}$$

$$\therefore \quad (ca'-c'a)^2=(bc'-b'c)(ab'-a'b).$$

Example 4. Shew that x^3+px^2+qx+1 and x^3+qx^2+px+1 cannot have a common factor, unless $p=q$ or $p+q+2=0$.

(i) Since the given expressions become identical if $p=q$, therefore each expression is a common factor, if $p=q$.

(ii) The common factor, if any, is obviously contained in the difference of these two expressions, *i.e.*, $(p-q)x^2+(q-p)x$, or is contained in $(p-q)x(x-1)$.

$(p-q)$ cannot be a common factor, since it is a constant.

and x is not a common factor, because each expression contains a term independent of x,

$\therefore$ $(x-1)$ can be the *only possible factor*.

Now, if $(x-1)$ be taken as the common factor, each expression must vanish if $x=1$, in either case

$$1+p+q+1=0$$

or

$$p+q+2=0.$$

Example 5. If $x+c$ be the H. C. F. of x^2+ax+b and $x^2+a'x+b'$, prove that their L. C. M. will be

$$x^3+(a+a'-c)x^2+(aa'-c^2)x+(a-c)(a'-c)c.$$

As $(x+c)$ is the H. C. F. of the given expressions, therefore each is exactly divisible by $x+c$.

Dividing each by $x+c$, we get $(x+a-c)$ and $(x+a'-c)$ as the quotients.

$$\text{L. C. M.} = (x+c)(x+a-c)(x+a'-c)$$
$$= x^3 + x^2\{c+(a-c)+(a'-c)\} + x\{c(a-c) + c(a'-c)+(a-c)(a'-c)\} + c(a-c)(a'-c).$$
$$= x^3 + x^2(a+a'-c) + x(aa'-c^2) + (a-c)(a'-c)c.$$

EXERCISE 105.

1. For what value of x will the expressions x^3-4x^2+2x+4 and $3x^3-7x^2+3x-2$ vanish?

2. Find the value of m, in order that x^2+mx+2 and x^2+5x+6 may have a common factor.

3. Find the expression which divides $x^4+x^3+4x^2+5x+8$ and $2x^4+4x^3+3x^2-2x-4$, leaving $2x-1$ and $4x+5$ as remainders respectively.

4. The H. C. F. and L. C. M. of two integral expressions, each of the second degree, are $x-4$ and $x^3-3x^2-10x+24$ respectively. Find the expressions.

5. If H be the H. C. F. and L the L. C. M. of two expressions A and B and if $H+L=A+B$, prove that $H^3+L^3=A^3+B^3$.

6. If $x+a$ be the H. C. F. of x^2+px+q and $x^2+p'x+q'$, shew that $a=\dfrac{q-q'}{p-p'}$.

7. Find the condition so that ax^3+bx+c and $a'x^3+b'x+c'$ may have a common factor of the form $x+m$.

8. For what value of a will the expressions $x^3-ax^2+19x-a-4$ and $x^3-(a+1)x^2+23x-a-7$ have a common factor?

9. Find the value of p so that x^2+x+p and x^3+x^2+x+1 may have a common linear factor.

10. Shew that ax^2+bx+c and cx^2+bx+a cannot have a common factor of the form $x+p$, unless $a+c=b$ or $a+b+c=0$.

11. If x^2+qx+r and x^3+px^2+qx+r have a common linear factor, then $(p-1)^2-q(p-1)+r=0$.

12. If x^2+ax+b and $x^2+a'x-b$ have a common linear factor, then $4b=a^2-a'^2$ and the factor is $x+\frac{1}{2}(a+a')$.

13. For what value of a, other than zero, will the expressions x^2+5x+a and x^3+3x+a have a common linear factor? *Hence*, find their H. C. F.

SECTIONAL REVISION VI

TEST PAPERS

PAPER 1

1. Exemplify the distinction (i) between an integral function and a fractional function; (ii) between a rational function and an irrational function.

Write down a general expression in x and y of the 3rd degree.

2. (i) Multiply, using the method of detached co-efficients, $5x^4-2x^2+x-4$ and $3x^3+x+2$.

(ii) Without actual multiplication find the value of a for which the co-efficient of x^2 in $(4x-1)(2x+3)(x-a)$ vanishes.

3. Under what conditions will the expression $9x^2+py^2+qz^2-4xy+ayz+bzx$ be symmetrical ?

4. If $x \equiv A(x-2)+B(x-1)$, find the values of A and B.

5. By the remainder theorem, find the remainder if $3x^5+2x^3-x^2+4x-1$ be divided by $x-2$.

6. Factorise $56x^2-59xy-231y^2$ by the application of $a+b=m$ and $ab=n^2$.

7. Find the equation whose roots are -4 and 5.

PAPER 2

1. (i) If $f(x)=4x^3-5x^2+3x-1$, find the value of $f(0)$, $f(1)$ and $f(-1)$.

(ii) Divide by the method of detached co-efficients $3x^3-2x+2-x^2$ by $3x^2+2-4x$.

2. Write down a homogeneous expression of the 4th degree in x, y and z.

3. Write down all the types of terms in the expression $\Sigma \frac{x^2(a-b)}{ab}$, where a, b, c are variables and x is constant.

4. If $a_0x^4+a_1x^3+a_2x^2+a_3x+a_4\equiv(x^2-2x+3)(x^2+x-1)$, find the values of a_0, a_1, a_2, a_3 and a_4.

5. Factorise $(2x-y)^5+(2y-z)^5+(z-y-2x)^5$.

6. Simplify $\Sigma\dfrac{a^2}{(a-b)(a-c)(x-a)}$ where a, b, c are variables and x is constant.

7. Find the equation whose roots are $+8$ and -7.

PAPER 3

1. Exemplify the distinction between absolute symmetry and cyclic symmetry.

2. Write down a cyclic expression in x, y, z whose first term is $\dfrac{x^2-y^2}{x(y-z)}$.

3. Without actual multiplication, state why $(a^2+b^2)(a+b)=a^3+2a^2b+3ab^2+b^3$ is wrong.

4. Express $\dfrac{x^2-2x-14}{(x-1)(x-2)(x+3)}$ in proper partial fractions with denominators of the first degree in x.

5. Factorise $(2x-3y+z)^3+(x+2y-z)^3+(y-3x)^3$.

6. Solve $x^2-6x-55=0$ by factors and also graphically by the *first method*.

PAPER 4

1. Write down $a^3(c-b)+b^3(a-c)+c^3(b-a)$ in cyclic order.

2. Express $\dfrac{5x^4-12x^3}{(x-1)(x-2)}$ in the form of

$$Ax^2+Bx+C+\frac{D}{x-1}+\frac{E}{x-2}.$$

3. A homogeneous and symmetrical expression of the 2nd degree in x, y and z has the value 14 when $x=-3$, $y=4$ and $z=3$ and the value 16 when $x=-2$, $y=1$, $z=-5$; find it.

4. Shew by the remainder theorem that x, $(x-1)$, $(x+1)$ are the factors of $x^4+2x^3-x^2-2x$.

5. Factorise $x^3-13x+12$ by the remainder theorem.

6. Use the method of cross-multiplication in solving the following equations :

$$\left.\begin{aligned} 3x-2y+z&=0\\ 2x+y-11z&=0\\ x+4y+3z&=26. \end{aligned}\right\}$$

7. Solve $2x-1=\frac{15}{x}$ by factors and also graphically by the *second method.*

PAPER 5

1. Find the values of A, B and p, in order that
$$A(x-1)^2+B(x-p)^2\equiv x^2-8x+10.$$

2. Shew that $p+q+r+s$ is a factor of
$$(p+r)(p+s)(q+r)(q+s)-(pq-rs)^2.$$

3. Simplify by cyclic symmetry $\Sigma(b+c-2a)(b+c-a)$ where a, b, c are variables.

4. If $x+y+z=0$, shew that $\Sigma(x-y)^2=3\Sigma x^2$ where x, y, z are variables.

5. Factorise $10x^4-63x^3+72x^2+63x+10$.

6. Solve the equation $12x^2+16x-35=0$ by factors and verify the solution.

7. Find the square root of $x^6-6x^5+17x^4-26x^3+22x^2-8x+1$ by the method of indeterminate co-efficients.

PAPER 6

1. Determine the values of a and b for which
(i) $3x^4-7x^3+9x^2+ax+b$ and (ii) ax^4+bx+3
are exactly divisible by $(x-1)^2$.

2. Shew that $l+m+n$ is a factor of
$l^2(m+n)+m^2(n+l)+n^2(l+m)$ if $l^3+m^3+n^3=0$.

3. Shew that $5^{3n}-4^{3n}$ is always divisible by 61.

4. Simplify $\Sigma\dfrac{1}{a(a+b)(a+c)}$.

5. If $2s=a+b+c$, shew that
(i) $s^2+(s-a)^2+(s-b)^2+(s-c)^2=a^2+b^2+c^2$.
(ii) $s^3-(s-a)^3-(s-b)^3-(s-c)^3=3abc$.
(iii) $s^2+(s-b)(s-c)+(s-c)(s-a)+(s-a)(s-b)$
$=bc+ca+ab$.

6. Factorise $8x^3+4x+3$.

7. Solve the equation $15x^2+26x-21=0$ by completing the square and verify the solution.

PAPER 7

1. Using the principle of indeterminate co-efficients, prove
$$(x+y+z)^3=x^3+y^3+z^3+3xy(x+y)+3yz(y+z)+3zx(z+x)+6xyz.$$

2. Factorise (i) $(a+b)^5-a^5-b^5$.
(ii) $\Sigma a^2(b^3-c^3)$.

3. Simplify $\Sigma \dfrac{a}{ab(a-b)(b-c)}$, where a, b, c are variables.

4. If $a+b+c=0$, shew that (i) $\Sigma a^3=3abc$,
(ii) $\Sigma a^5=-5abc\Sigma ab$

where a, b, c are variables.

5. For what value of x will the expression $2x^3+10x^2+14x+6$ and $x^3+x^2+7x+39$ vanish?

6. The H.C.F. and L.C.M. of two integral expressions, each of the second degree, are $x+2y$ and $6x^3+7x^2y-16xy^2-12y^3$ respectively; find them.

7. Solve the equation $3x^2-17x+10=0$ by the formula.

PAPER 8

1. Factorise (i) $x^2+(a-1)x-(2a-3)(a-2)$.
(ii) $2a^2-ab-3ac+7bc-3b^2-2c^2$.

2. If $x-p$ be the H.C.F. of x^2+ax+b and x^2+mx+n, then their L. C. M. is $x^3+x^2(a+m+p)+x(am-p^2)-p(a+p)(m+p)$.

3. Find the value of p so that $x^2+(p-3)x-3p$ and $x^3+8x^2+17x+10$ may have a common linear factor.

4. Solve the equations:
$$\left.\begin{aligned} x+y+z&=0\\ (b+c)x+(c+a)y+(a+b)z&=0\\ bcx+cay+abz&=1.\end{aligned}\right\}$$

5. Simplify $\dfrac{(x-b)(x-c)}{(a-b)(a-c)}$ + two similar terms.

6. Express in partial fractions $\dfrac{7x^2+x+1}{x^3-1}$.

7. Solve the equation $2x^2-13x+15=0$ graphically by *the third method.*

MISCELLANEOUS EXERCISES

[Selected Questions]

1. (*a*) Write down *briefly* $53{\cdot}7 \times 53{\cdot}7 \times 53{\cdot}7 \times 1000000$.
(*b*) Find the value of $\sqrt{s(s-a)(s-b)(s-c)}$ when $a=41$, $b=40$, $c=9$ and $2s=a+b+c$.

2. (*a*) Multiply $3x^4-2x^3+x-4$ and $2x^3-5x^2+1$ by the method of detached co-efficients.

(*b*) Without doing full process, find the co-efficient of x^3 in the product of $4x^3-2x^2+3x-1$ and $2x^3+4x^2-x+3$.

3. Find the H.C.F. of x^5-x^3+8 and x^5-x^2+4.

4. (i) $2b=a+c$ and $\frac{2}{c}=\frac{1}{b}+\frac{1}{d}$, ; shew that $a:b=c:d$.

(ii) If a, b, c are in continued proportion, prove that $a^2+ab : b^2 :: b^2+bc : c^2$.

5. Resolve into factors:
(i) $a^2-b^2+ab(b-a)$.
(ii) $c(a^2+b^2-c^2)+b(c^2+a^2-b^2)+a(b^2+c^2-a^2)+6abc$.
(iii) $\left(a+\frac{1}{a}\right)^2-\left(b+\frac{1}{b}\right)^2$.

6. Simplify $\frac{(x+1)^2}{(x-y)(x-z)}$ + two similar terms.

7. Solve the equations (i) $x+y=8, y+z=10, z+x=12$.

(ii) $\frac{x-a}{b+c}+\frac{x-b}{c+a}+\frac{x-c}{a+b}=3$.

8. If $a=\frac{b}{1-b}$ and $b=\frac{c}{1-c}$, prove that $c=\frac{a}{1+2a}$.

9. Express $\frac{3x+11}{(x+2)(x+3)}$ in the form of $\frac{A}{x+2}+\frac{B}{x+3}$.

10. The perimeter of a rectangular field is 72 ft.; find its dimensions if it contains 315 sq. ft.

11. (*a*) Write down *briefly* $\frac{3{\cdot}4\times3{\cdot}4\times3{\cdot}4}{1{,}000{,}000}$.

(*b*) Find the value of $\sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$ when $a=10$, $b=21$, $c=17$ and $s=\frac{1}{2}(a+b+c)$.

12. (*a*) Divide $8x^5+22x^4-11x^3-9x^2+7x-2$ by $4x^2+x-2$ by the method of detached co-efficients.

(*b*) For what value of m will $2x^3-5x^2+mx-21$ be exactly divisible by $x-3$?

13. Find the H.C.F. of $x^4+9x-20$ and $5x^4+9x^3-64$.

14. If $a:b::b:c$, prove that

(i) $a-2b+c=\frac{(a-b)^2}{a}=\frac{(b-c)^2}{c}$.

(ii) $(a+b+c):(a-b+c)=(a+b+c)^2:(a^2+b^2+c^2)$.

15. Resolve into factors:

(i) $a^2b^2-a^2-b^2+1$.

(ii) $x(y^2-a)-y(x^2-a)$.

(iii) $324x^4+1$, and *hence* find out the factors of 3240001.

16. Simplify $\frac{bc(x-a)^2}{(a-b)(a-c)}$ + two similar terms.

17. Solve the equations (i) $\frac{x-1}{2}+\frac{2x-7}{3}=x-2$.

(ii) $\sqrt{4x+5}+\sqrt{4x-11}=8$.

18. Eliminate h, k and l from the equations:

$h=k^a$, $l=k^b$, $h^bl^a=\sqrt[c]{k^2}$.

19. Express $\frac{11x+5}{(x-2)(x+1)^2}$ in the form of $\frac{A}{x-2}+\frac{Bx+C}{(x+1)^2}$.

20. The difference between the length and breadth of a rectangle is 18 ft. and its area is 448 sq. ft.; find

(i) its semi-perimeter,

(ii) its length and breadth.

21. Find the value of:

(i) $128^{\frac{4}{7}}$, $\sqrt[3]{81^3}$ and $\left(\frac{1}{27}\right)^{-\frac{4}{3}}$.

(ii) $\left\{(a+b)^{\frac{1}{2}}+(a-b)^{\frac{1}{2}}\right\}^2$.

22. Shew that (i) 7^n-1 is divisible by 6, and

(ii) $45^{2n+1}+1$ is divisible by 46,

if n is a positive integer.

23. Find the H. C. F. of $2x^5-11x^2-9$ and $4x^5+11x^4+81$, by the method of alternate destruction of the highest and the lowest terms.

24. If $x:a=y:b=z:c$, then

$$\frac{x^3}{a^2}+\frac{y^3}{b^2}+\frac{z^3}{c^2}=\frac{(x+y+z)^3}{(a+b+c)^2}.$$

25. Resolve into factors:

(i) $x^2-x-1332$.

(ii) $z(x+y)^2-y(z+x)^2$.

26. Simplify $\dfrac{3\sqrt{3}-2\sqrt{2}}{\sqrt{3}-\sqrt{2}}$.

27. Solve the equations (i) $\dfrac{2}{2x-5}+\dfrac{1}{x-3}=\dfrac{6}{3x-1}$.

(ii) $32^x=\dfrac{8^x}{16}$.

28. Eliminate t from the equations $x=t+\dfrac{1}{t}$, $y^2=t^2-\dfrac{1}{t^2}$.

29. Find the values of A, B, C in the following identity:

$$\frac{x^2}{(x-1)(x-2)(x-3)}\equiv\frac{A}{x-1}+\frac{B}{x-2}+\frac{C}{x-3}.$$

30. A certain sum was divided equally among a certain number of persons; had there been 3 persons more, each would have received 2 as. less, and had there been 6 persons fewer, each would have received 6 as. more; find the sum of money and the number of them.

31. (*a*) Find the value $\sqrt[5]{243^3}$ and $1296^{-\frac{3}{4}}$.

(*b*) If $x+\frac{1}{y}=1$, $y-\frac{1}{z}=1$, prove that $xyz=1$.

32. (*a*) Shew that $5^{3x}-4^{3x}$ is always divisible by 61.

(*b*) Divide $2x^3+5x^2-mx+4$ by x^2+2x-1, and find the value of m for which the given divisor will be a factor of the given dividend.

33. Find the H. C. F. of $8x^4+3x+10$ and $10x^4+3x^3+8$.

34. If $\frac{a^2+c^2}{ab+cd}=\frac{ab+cd}{b^2+d^2}$, shew that $\frac{a}{b}=\frac{c}{d}$.

35. Resolve into factors:

(i) $a(a+1)x^2+x-a(a-1)$.

(ii) $x^4+y^4+1+2x^2y^2-2x^2-2y^2$.

36. Simplify $\frac{x^3+6x^2+2x-15}{x^3+5x^2-2x-10}$.

37. Solve the equations:

(i) $\left.\begin{aligned}3^{x+1}+2^y&=85\\3^x-2^{y+2}&=11.\end{aligned}\right\}$

(ii) $\left.\begin{aligned}x+y+z&=3x+2y+z=0\\2x-3y+4z&=24.\end{aligned}\right\}$

38. Prove that $\frac{n(n-1)(n-2)}{1.2.3}+\frac{n(n-1)}{1.2}=\frac{(n+1)n(n-1)}{1.2.3}$.

39. If $\frac{x^2+3x+7}{(x^2+x+1)(x+1)^2}\equiv\frac{A}{x+1}+\frac{B}{(x+1)^2}+\frac{Cx+D}{x^2+x+1}$, find the values of A, B, C and D.

40. A person bought 13 horses and 9 cows for Rs. 4,185 and at the same rates 8 horses and 6 cows for Rs. 2,640. Find the price of 9 horses and 5 cows.

41. (i) If $2s=a+b+c$, prove that

$16s(s-a)(s-b)(s-c)=(a+b+c)(b+c-a)(c+a-b)(a+b-c)$.

(ii) If $a=AP^{p-1}$, $b=AQ^{q-1}$, $c=AR^{r-1}$, then

$$a^{q-r}.\ b^{r-p}.\ c^{p-q}=1.$$

42. (a) Simplify by the shortest possible method.

$$(5x-8)^3-(3x-8)^3-6x(5x-8)(3x-8).$$

(b) Without actual multiplication, work out the co-efficient of x^3 in the product of $x^4-5x^3-2x^2+7x-4$ and x^2-3x+1.

43. Find the H.C.F. of $3x^5-5x^3+2$ and $2x^5-5x^2+3$, by the method of the alternate destruction of the highest and the lowest terms.

44. If $\frac{1}{b+c}+\frac{1}{c+a}=\frac{2}{a+b}$, shew that $a^2+b^2=2c^2$.

45. Resolve into factors:

(i) x^3+64.

(ii) $(x+y+z)(xy+yz+zx)-xyz$.

46. Simplify $\dfrac{3^{2x+1}.\ 5^{2x-y}.\ 7^{2x+y+2}.\ 15^y}{15^x.21^{y+x+2}.35^x}$

47. Solve the equations (i) $\frac{x-1}{x-2}+\frac{x-5}{x-6}=\frac{x-?}{x-3}+\frac{x-4}{x-5}$.

(ii) $y-\frac{1}{x}=x-\frac{1}{y}=\frac{3}{2}$.

48. Eliminate x, y, z from $y=az$, $z=bx$, $x=cy$.

49. Find the values of A, B, C in $(2x-3)^2\equiv Ax^2+2Bx-3C$.

50. A railway journey of 240 miles would take half an hour less if the speed of the train were increased by 12 miles an hour. Find the speed in miles per hour.

51. (i) If $2s=a+b+c$, shew that

$$(s-a)^3+(s-b)^3+(s-c)^3+3abc=s^3.$$

(ii) If $a^x=b^y=c^z$ and $ac=b^2$, then $\frac{1}{x}+\frac{1}{z}=\frac{2}{y}$.

52. (i) For what value of k is $36x^4-36x^3+kx^2-12x+4$ a perfect square?

(ii) Fill in the blanks in the following:

$$\frac{x}{5}=\frac{y}{3}=\frac{4x+3y}{\quad}=\frac{3x-2y}{\quad}.$$

53. Extract the square root of:

$$x^4+4x^2+\frac{1}{x^2}-2x-\frac{4}{x}+4.$$

54. If $a=\sqrt{\frac{x-1}{x+1}}$, express $\frac{1-a}{1+a}$ in terms of x.

55. Resolve into factors:

(i) x^4+64.

(ii) $x^2+(a+b+c)x+ab+ac$.

56. Simplify $\Sigma\frac{1}{(a-b)(a-c)(x-a)}$.

57. Solve the equations (i) $3^x+3^{-x}=9\frac{1}{9}$.

(ii) $a^{2x^m}=\{(a^x)^m)\}^x$.

58. Draw the graph of $\frac{3x-7}{6}$. From the graph find the value of the function when $x=3{\cdot}5$; also find for what value of x the function becomes equal to $1{\cdot}1$?

59. Determine the values of p, q, r in

$$(x-1)(x-2)(x-3)\equiv(x-4)^3+p(x-4)^2+q(x-4)+r.$$

60. A farmer bought an equal number of two kinds of sheep, one kind @ Rs. 6 each and the other @ Rs. 8 each; if he had spent his money equally on the two kinds, he would have had 3 sheep more than he had. How many of each kind did he buy?

61. (*a*) Write down *briefly* the value of

$$\frac{52{\cdot}9 \times 52{\cdot}9 \times 52{\cdot}9 \times 1000000}{125(3{\cdot}6)^3}.$$

(*b*) Find the value of $\dfrac{5^{x+1}}{(5^x)^{x-1}} \div \dfrac{25^{x+1}}{(5^{x-1})^{x+1}}$.

62. (i) If $a+b+c=15$ and $ab+ac+bc=71$, find the value of $a^2+b^2+c^2$.

(ii) If $a^2+b^2+c^2=155$ and $ab+ac+bc=143$, find the value of $a+b+c$.

63. The H.C.F. and L.C.M. of two integral expressions A and B, each of the second degree, are $x+3$ and x^3-7x+6; find A and B.

64. If $a:b::b:c$ shew that

$$(a+b+c)(a-b+c)=a^2+b^2+c^2.$$

65. Resolve into factors :

(i) $16x^8-1$.

(ii) $(5x+3y)^3-(3x+5y)^3-8(x-y)^3$.

66. Simplify $\dfrac{x^3-1}{x^2-1} \times \dfrac{x^2+3x+2}{x^4+x^2+1} \div \dfrac{x^2+x-2}{x^4+x}$.

67. Solve the equations (i) $\left.\begin{aligned}\frac{1}{x}+\frac{\cdot 1}{y}&=\cdot 25\\ \frac{1}{y}+\frac{\cdot 1}{x}&=\cdot 52\end{aligned}\right\}$

(ii) $\sqrt{3x+2}-\sqrt{3x-10}=2$.

68. If $x-\dfrac{1}{x}=k$, find the value of $x^3-\dfrac{1}{x^3}$.

69. Find the values of k, l, m in

$$2x^2-11x+5\equiv(kx+l)(x-3)+m(x^2+2x-5).$$

70. A number consists of two digits. When the number is divided by the sum of its digits, the quotient is 7. The sum of the reciprocals of the digits is 9 times the reciprocal of the product of the digits. Find the number.

71. If $x=b+c$, $y=c+a$ and $z=a+b$, find the value of

$$\frac{x^3+y^3+z^3-3xyz}{a^3+b^3+c^3-3abc}.$$

72. (*a*) For what value of x will both the expressions x^3-x^2-5x-3 and $x^3-4x^2-11x-6$ vanish?

(*b*) To make $9x^4-12x^3+10x^2-3x-3$ a perfect square, (i) what should be added to it, (ii) what should be subtracted from it, (iii) what values should be given to x?

73. The H.C.F. of two expressions is x^2+3x+2 and their L.C.M. is $(x^2+3x+2)(x-3)(x+5)$, one expression is $x^3+8x^2+17x+10$; find the other.

74. (i) If $ax=y$ and $by=x$, shew that $\frac{1}{1+a}+\frac{1}{1+b}=1$.

(ii) $a:b=c:d=e:f$, prove that

$$(a^2+c^2+e^2)(b^2+d^2+f^2)=(ab+cd+ef)^2.$$

75. If $x+y+z=6$ and $xy+yz+zx=9$, shew that

$$\frac{1}{1-x}+\frac{1}{1-y}+\frac{1}{1-z}=0.$$

76. Simplify $\dfrac{(x^2)^{a+b}\times(x^2)^{b+c}\times(x^2)^{c+a}}{x^{4a}\times x^{4b}\times x^{4c}}$.

77. Solve the equations (i) $\dfrac{x+\sqrt{x-1}}{x-\sqrt{x-1}}=\dfrac{3}{7}$

(ii) $x^2-y^2=18$, $x-y=3$.

78. Shew that $N(N^4-1)$ will be divisible by 48, if N is an odd number.

79. If $f(x)=x^3-3x^2+3x-1$ find the values of $f(0)$, $f(1)$ and $f(x+1)-f(x-1)$.

80. A man, who went out for a walk between 5 and 6 and returned between 6 and 7, found that the hands of his watch had exactly changed places. When did he go out?

[*Hint.* Suppose he went out at x minutes past 5 and returned at y minutes past 6. From these suppositions, we get

$$x=30+\frac{y}{12} \text{ and } y=25+\frac{x}{12}.]$$

81. (*a*) If $x^2+y^2=1$, complete $\frac{x}{a}=\frac{y}{b}=\frac{1}{\quad}$.

(*b*) Fill in the blank in $\frac{x^2+y^2}{a}=\frac{xy}{b}=\frac{x^2-y^2}{\quad\quad}$.

82. (*a*) Find the value of $x^3+y^3-z^3+3xyz$ when $x=3{\cdot}461$, $y=2{\cdot}314$, $z=5{\cdot}775$.

(*b*) Find the continued product of

$$x^2+x+1,\ x^2-x+1,\ x^4-x^2+1.$$

83. For what value of x will the expression $2x^4-7x^3+13x^2-9x-3$ be divisible by x^2-2x+3?

84. If $x+y=a$ and $xy=b$, express x^4+y^4 in terms of a and b.

85. Simplify $\frac{x^3+1}{x^2-5x+6}\times\frac{x^2-7x+10}{x^4+x^2+1}\div\frac{x^2-4x-5}{x^3+x^2+x}$.

86. Solve the equation $\frac{1}{x-1}+\frac{4}{x-4}=\frac{2}{x-2}+\frac{3}{x-3}$.

87. If $\frac{x}{a-b}=\frac{y}{b-c}=\frac{z}{c-a}$, prove that $x(a+b)+y(b+c)+z(c+a)=0$.

88. Construct a homogeneous and symmetrical expression of the second degree in x and y, which is equal to 128 when $x=-y=4$ and which is equal to 120 when $x=1$, $y=-5$.

89. Simplify $(a+b+c)^4-(b+c)^4-(c+a)^4-(a+b)^4+a^4+b^4+c^4$ by the principles of homogeneity, symmetry and indeterminate co-efficients.

90. An officer arranged his army of 3,900 soldiers in a hollow square 15 deep; find the number of soldiers in the front row.

91. (*a*) Simplify $\left(\frac{x^a}{x^b}\right)^{a+b} \times \left(\frac{x^b}{x^c}\right)^{b+c} \times \left(\frac{x^c}{x^a}\right)^{c+a}$.

(*b*) Shew that $7^{2n+1}+1$ is exactly divisible by 8 for all integral values of n, even or odd.

92. (*a*) If $x=b+c-a$, $y=c+a-b$, $z=a+b-c$, then

$$x^3+y^3+z^3-3xyz=4(a^3+b^3+c^3-3abc).$$

(*b*) Find the continued product of
$(1+x)(1+x^2)(1+x^4)$, without actual multiplication.

93. (*a*) Shew that an integral expression is divisible by $x-1$ if the sum of its co-efficients is zero.

(*b*) Fill in the blanks
$(x-a)(x+b)(x+\quad\quad)\equiv x^3+x^2(\quad\quad)+x(\quad\quad)-abc.$

94. Find the L.C.M. of $3a^2-ab-2b^2$, $8a^3-27b^3$ and $6a^2-5ab-6b^2$.

95. Resolve into factors (i) $a^3-19ab^2+30b^3$,
(ii) $a^3-17x+26$.

96. Simplify $\dfrac{2^6\times 16^{-2}\times 256}{4^{-9}\times 2^{23}}$.

97. Find the square root of $22-12\sqrt{2}$.

98. If $a+b+c=0$, shew that
$$a^2+ab+b^2=b^2+bc+c^2=c^2+ac+a^2.$$

99. Prove that $(a+b+c)^3-a^3-b^3-c^3\equiv 3(a+b)(b+c)(c+a)$.

100. An officer can form his men into a hollow square 5 deep and also into a hollow square 6 deep, but the front in the latter case contains 4 men fewer than in the former. Find the number of men.

ANSWERS

[*Answers to the oral exercises have not been given intentionally.*]

EXERCISE 2. (Page 7.)

1. (i) 7^4, (ii) a^5, (iii) ap^3, (iv) ab^2, (v) $(\cdot01)^3$, (vi) $\frac{1}{a^3}$;
(vii) $3m^2+4n^2$, (viii) a^2n^3, (ix) p^4+p^3, (x) ax^3+by^3.
3. (i) 625, (ii) 49, (iii) 32, (iv) 81. 5. (i) 129, (ii) 359, (iii) 1113, (iv) 1114, (v) 14, (vi) 67. 6. 40, 65, 120, 40, 85.
8. (i) 5^7 or 78125, (ii) 7^6 or 117649, (iii) a^{13}, (iv) x^{13}.
9. (i) $6abcd$, (ii) $12x^4y^3$, (iii) $a^3b^3c^2$, (iv) $a^4b^2c^2$, (v) $15a^5b^7c^3$, (vi) $6p^5q^4$, (vii) $60m^4n^3p^5$. 11. (i) a^5, (ii) p^9, (iii) m^6, (iv) x^{12}. 12. (i) $2x^2y^4$, (ii) $5m^3n$, (iii) $2p^2q^2$, (iv) $2m^4p$, (v) $3l^2n^7$. (vi) $6x^3y^4z^3w^2$. 14. (i) 2^6 or 64, (ii) 3^6 or 729, (iii) a^{3m}, (iv) x^{3m}, (v) p^{2n}, (vi) y^{15}. 16. (i) 5^2 or 25, (ii) 4^3 or 64, (iii) x^6, (iv) $2x^3$, (v) $3y^4$, (vi) $5ab^2$. 17. (i) $4a^3$, (ii) $6x^4$, (iii) $5y^9$. 18. (i) $2a^2$, (ii) $3b^3$, (iii) $4c^3$. 19. 22.
20. 432.

EXERCISE 3. (Page 12.)

1. $11a^2$. 2. $9ab$. 3. $15p$. 4. $28x^7$.
5. $6ab$. 6. $5abc$. 7. $5a^2$. 8. $4x^2$.
9. (i) $12a$, (ii) 12×13. 10. (i) $6x$, (ii) 6×5.
11. (i) $14x$, (ii) 14×4. 12. (i) $14y$, (ii) 14×7.
13. $2a+2b$. 14. $2x+3y$. 15. $2a+2b$. 16. $3x+2y$.
17. $3x+7$. 18. $2x+y+4$. 19. $a+b$.
20. $3x+7y+4$. 21. $2a$. 22. $3a+2b-2c$.
23. $2xy+3xz$. 24. $6a^2-3a$ 25. x^3+x^2+4x.
26. $2p^2+p+3$. 27. a^4+4a^2+1.
28. $10+a-2a^2$. 29. a^3+2a^2+2a+1.
30. $7x^3-3x^2-2x+4$.
31. (i) $7x^4+x^3+5x^2+3x+4$, (ii) $4+3x+5x^2+x^3+7x^4$.
32. (i) $a^7+4a^6+5a^3+a^2+1$, (ii) $1+a^2+5a^3+4a^5+a^7$.
33. $5+3x+2x^2+2x^3+x^4$. 34. $3x^4-4x^3+8x^2+4x+9$.
35. $3.10^2+7.10+4$; no. 36. $5.10^3+7.10^2+4$; no.

EXERCISE 4. (Page 15.)

(i) $2a^2b^2$, $12a^3b^3$; (ii) $6ab$, $36a^3b^3$; (iii) $3ab$, $12a^2b$; (iv) $13a^2b^4$, $156a^3b^5$; (v) ab, $12a^3b^3c^2$; (vi) $2ab$, $72a^2b^2c^2$.

3. (i) 2.7^2, $2^3.7^3$ or 98, 2744; (ii) $3^2.5^2$, $3^3.5^3$ or 225, 3375; (iii) 144, 4320; (iv) 210, 4200.

4. (i) $\frac{3bc}{5a}$, (ii) $\frac{3xa}{y}$, (iii) $\frac{4y}{x}$, (iv) $\frac{1}{3acb}$, (v) $3xyz$. (vi) $\frac{3x}{ay^2}$. 5. The gaps are: (i) $35xy$, (ii) $12a^3b$, $2a$. (iii) a^3b^2, (iv) ab^2, (v) amn^3.

6. (i) $\frac{5x}{6ax}$, $\frac{8a}{6ax}$, $\frac{18c}{6ax}$; (ii) $\frac{9xy^2}{12x^2y^2}$, $\frac{2x^3}{12x^2y^2}$, $\frac{4y^3}{12x^2y^2}$; (iii) $\frac{6cx^2}{12abc}$, $\frac{4by^2}{12abc}$, $\frac{3az^2}{12abc}$; (iv) $\frac{60b^3x^2y}{60a^3b^3}$, $\frac{40ab^2x^3}{60a^3b^3}$, $\frac{45a^2by^3}{60a^3b^3}$, $\frac{48a^3xy^2}{60a^3b^3}$. 7. (i) $\frac{17x}{12}$, (ii) $\frac{3a+2b}{2x}$, (iii) $\frac{5}{6x}$, (iv) $\frac{x^2+y^2}{xy}$, (v) $\frac{x+y}{y}$, (vi) $\frac{1+3x}{x}$, (vii) $\frac{2}{3x}$, (viii) $\frac{a}{3b}$, (ix) $\frac{b-a}{b}$, (x) $\frac{xy-1}{xy}$, (xi) $\frac{z-x^2y^2}{xyz}$, (xii) $\frac{13x}{12}$, (xiii) $\frac{bcx+acy+abz}{abc}$, (xiv) $\frac{61x}{60}$, (xv) $\frac{13a}{12x}$, (xvi) $\frac{8ayz+5bxz+9cxy}{12xyz}$, (xvii) $\frac{15x^2-36xy+20y^2}{180}$, (xviii) $\frac{a^2+b^2+c^2}{abc}$, (xix) 0.

8. (i) $\frac{ab}{c^2}$, (ii) z^2, (iii) 1, (iv) abc, (v) $\frac{6c}{25a}$, (vi) $\frac{8x}{9z}$, 9. $\frac{1}{3}$.

10. No. 11. (i) $\frac{x}{x+y}$, (ii) $\frac{x+y}{y}$.

EXERCISE 5. (Page 18.)

6. a exceeds b by $a-b$. 7. $a+b=b+a$. 8. $a\times b=b\times a$.

9. $a^2\times a=a^3$. 10. $a\times\frac{1}{a}=1$. 11. $0\times a=0$. 12. $a\times 0=0$.

13. $2n$ is an even number. 14. $2n+1$ is an odd number.

15. The sum of the first n odd numbers is n^2.

16 $4x+x=5x$. 17. $m\%$ of $x=\frac{mx}{100}$ of x. 18. If the length of a rectangle is l ft. and breadth b ft., its area $=l.b$ sq. ft.

19. $x-1$, x and $x+1$ are three consecutive numbers.

20. (a) $p+1$, (b) $p-1$. 21. $m-1$, $m+1$. 22. 120 sq. ft.

23. $707\frac{1}{7}$ sq. ft. 24. Rs. 81. 26. 104 degrees.

27. (i) 6 rt. angles, (ii) 12 rt. angles, (iii) 16 rt. angles, (iv) 20 rt. angles. 28. (i) $\frac{4}{3}$ rt. angles, (ii) $\frac{3}{2}$ rt. angles.

29. $(16x+y)$ annas. 30. (i) $\frac{x}{5}$ hours, (ii) $\frac{x}{y}$ hours.

31. (i) $\frac{mx}{l}$ miles, (ii) $\frac{440m}{t}$ yards. 32. $\frac{88m}{3}$ yards.

33. $\frac{60x}{s}$ days. 34. 40 shillings, $\frac{8x}{5}$ shillings. 35. $\frac{4x}{3}$ miles.

36. $\frac{ax}{15}$ yds. 37. (i) Rs. $3y$, (ii) Rs. $\frac{my}{x}$. 38. $\frac{7}{3}$ days.

$\frac{14x}{y}$ days. 39. $\frac{xy}{x+y}$ days. 40. $\frac{15(16x+y)}{74}$ srs. 41. Rs. $\frac{n-m}{m}$.

42. $\frac{l+m+n}{a+b+c}$ miles per hour. 43. $\frac{29x}{25}$. 44. $\frac{100x}{y}$. 45. (i) $\frac{100}{a}$,

(ii) $\frac{100x}{b}$. 46. Rs. 15r. 47. Rs. $\frac{pr}{20}$. 48. Rs. $\frac{pri}{100}$.

49. $\frac{100}{x}$ years. 50. $\frac{4x}{25}$ seeds. 51. $\frac{27a}{4}$ annas. 52. $\frac{19r}{20}$ rupees.

53. $\frac{4x}{25}$ as. 54. $\frac{3(100-p)}{25}$ seers per rupee. 55. $15l$ sq. ft.

56. Rs. $\frac{xp}{4}$. 57. $\frac{A}{3}$ ft. 58. $\frac{9xy}{20}$.

59. abx cubic inches, $2(ab+ac+bc)$ sq. inches.

60. $\frac{x}{lb}$ ft. 61. $30x$ cubic inches. 62. $\frac{32xy}{9}$.

63. $\frac{320ab}{9}$ 64. $\frac{64b.w(l+b)}{27}$. 65 hk.

66. 1445 ft.

EXERCISE 8. (Page 31.)

3. 6 divisions in the negative direction.
4. 3 divisions in the positive direction. 5. 3 divisions in the positive direction. 6. 13 divisions in the positive direction.
7. 5 divisions in the negative direction. 8. 14 divisions in the negative direction. 9. 12 divisions in the positive direction.
10. 6 divisions in the negative direction.
11. 12 divisions in the negative direction.
12. $+$Rs. 60 $-$ Rs. 25 $-$ Rs. 15 $+$ Rs. 12 $-$ Rs. 16 $+$ Rs. 24 $+$ Rs. 40.
13. $-6x+8x-4x+2x+3x-2x-5x$.
14. $+8x-5x+11x-6x+13x-7x+15x$. 15. $-10a+15a-8a+16a$
16. $+$850 ft. $-$ 350 ft. $+$ 260 ft. $-$ 145 ft. 17. $-5°, -2°, +4°, +8°$.
18. (i) $+6$ hrs., (ii) $+2$ hrs., (iii) $-2\cdot5$ hrs., (iv) $+5$ hrs. 20 mts. 19. (i) $+46°$, (ii) $-8°$, (iii) $0°$, (iv) $-32°$.
20. (i) $+15$ mls., (ii) -15 mls., (iii) -20 mls., (iv) -35 mls.

EXERCISE 9. (Page 36.)

11. $-2ab$. 12. $14xyz$. 13. $-4abx^2$. 14. $-18pq^2r$
15. $-a+4b-2c$. 16. $-3x-z$. 17. $-4b^2+2pq+9q^2$
18. $3ab+4bc+4ca$. 19. $2x^2yz+4xy^2z+2xyz^2$.
20. $12ax^2y-3a^2xy-11axy^2$. 21. (i) 20_r (ii) 3_r (iii) -14_r
(iv) -7, (v) $-\frac{1}{2}$, (vi) 0. 22. $+2, +6, +2, -4, +1, -5$.
23. $+2, +8, -10, 0, -9, +1, +3, -6, +4, +9$.
24. x^2+xy+y^2. 25. x^2+y^2 26. a^2-b^2.
27. $-4p^2+3pq+3q^2$. 28. $3a^2-4a-4$.
29. $-6x^2+4xy-5y^2$. 30. $-6x-2y-5z$.
31. $-3b^2+b^2c-4bc^2$. 32. $a^2x^2-2ax-7$. 33. $3a^2-8b^2$
34. $-2p^2-4pq+11q^2$. 35. $-2+x^2+4x^3$.
36. $-a+4c-19d+5e+6f$. 37. $x^2+y^2+3x-4xy+2yz-9$.
38. $1\frac{9}{12}a^2+\frac{7}{3}ab+\frac{1}{6}b^2$. 39. $-x+12$. 40. $-a$.

EXERCISE 10. (Page 41.)

29. $2x+2z$. 30. $-3ab+7bx-ac$. 31. $-6y^2+4xy$.
32. $-x^2-2y^2-18$. 33. $2-m^2-m^3-m^6$. 34. $3x^2-4y^2-1$.
35. $-7a^2+3ab-12b^2$. 36. $-4x^2y^2-y^2z^2+8z^2x^2$.
37. $p^3-5p^2q+4q^3-6$. 38. $2x^2+\frac{1}{4}xy+2y^2-3z^2$.

39. a^2+b^2. 40. x^2+y^2-2xy. 41. $3a^2-2b^2-5$.
42. $3b^2-a^2-3$. 43. $-2a^2-b^2+8$. 44. $2a^2+3b^2-4$.
45. $4a^2-3b^2+2$. 46. $-2a^2+b^2+12$.
47. (i) -7 is greater than -12 by 5, (ii) 0 is greater than -4 by 4.

EXERCISE 11. (Page 44.)

5. $-p-3q$. 6. $2a-b$. 7. $2p+q$. 8. $-p+3q$.
9. $11k+2l$. 10. $5a-9b$. 11. $6l-9m+5n$.
12. $-2c$. 13. $-4p+3q+1$. 14. $p+2q+1$. 15. $4a-2b$.
16. $2a$. 17. $6y-3x-6z$. 18. $+1$. 19. -4. 20. 0. 21. 1.
22. $2b$. 23. $2y-2x-2z$. 24. 0. 25. 0. 26. $-2b$.
27. $5-(3-9)$.
28. (i) $-a^2+(m^2+b^2-m^2+b^2+c^2)$,
$-a^2-(-m^2-b^2+m^2-b^2-c^2)$, (ii) $x^2+(-a^2+bc-b^2-ca+c^2-ab)$,
$x^2-(a^2-bc+b^2+ca-c^2+ab)$. 29. (i) $(2x^3-5x^3)+(x^3-4x^3)$
$+(3x^2+x^2)$, $-(5x^3-2x^3)-(4x^3-x^3)-(3x^2-x^2)$,
(ii) $(ax^3-dx^3)+(-bx^2-ex^2)+(cx-fx)$,
$-(dx^3-ax^3)-(ex^2+bx^2)-(fx-cx)$.

30. (i) -1, (ii) $+a$.

EXERCISE 13. (Page 53.)

4. (i) -1, (ii) $+1$, (iii) $-a^{83}$, (iv) a^{94}, (v) b^{43},
(vi) m^{54}, (vii) n^{71}.
5. 0. 6. 0. 7. $-49, 63, -21, 3, -3, -1$. 8. (i) -50,
121, -11, 1; (ii) $-42, -197, 3, 4$; (iii) $3, -117, -13, -9$.
9. (i) $648y^{12}$, (ii) $-4a^{14}b^7$. 10. $4x^4-18x^3y+10x^2y^2-xy^3+y^4$.

EXERCISE 14. (Page 57.)

2. x^2+5x+6. 3. $x^2+7x+10$. 4. $b^2+ab+bc+ac$.
5. $a^2+2ab+b^2$. 6. $6a^2+7ab+2b^2$. 7. $2x^2+5xy+3y^2$.
8. $a^3b-2a^2b^2+ab^3$. 9. $-x^6+20x^3-100$.
10. $-3x^3y+7x^2y^2-2xy^3$. 11. $-x^3-2x^2y-xy^2$.
12. $4x^4-25a^2b^2$. 13. $-9m^3n+9m^2n^2-2mn^3$.
14. $\frac{5}{4}x^2-1\frac{2}{5}\frac{1}{0}x+\frac{3}{2}$. 15. $-10a^3+\frac{2}{3}a^2-\frac{5}{4}a+1\frac{1}{2}$. 16. $1-a^3$.
17. $1+a^3$. 18. $a^3+2a^2b+2ab^2+b^3$. 19. $a^3-2a^2b+2ab^2-b^3$.
20. a^6-b^6. 21. a^6+b^6. 22. $21a^4-36a^3b-a^2b^2+48ab^3-36b^4$.
23. $\frac{3}{4}x^3-\frac{2}{3}xy^2+\frac{2}{5}y^3$. 24. $12x^4+24x^3y-25x^2y^2+29xy^3-12y^4$.
25. $21x^5-34x^4y+18x^3y^2-18x^2y^3+17xy^4-4y^5$.

26. $x^4+x^2y^2+y^4$. **27.** $x^4+x^3y+x^3z-3x^2yz+xy^3+xz^3-3xy^2z-3xyz^2+y^4+z^4+yz^3+y^3z$.

28. $x^4+x^3y-x^3z+3x^2yz+xy^3-xz^3-3xyz^2+3xy^2z+y^4+z^4-yz^3-y^3z$.

29. $2x^5-5x^4y+5x^3y^2-5x^2y^3+5xy^4-3y^5$.

30. $x^4-4x^3y+6x^2y^2-4xy^3+y^4-z^4$.

31. $a^5-5a^4+10a^3-10a^2+5a-1$. **32.** $1-5a^2+4a^4$. **33.** $1-a^5$.

34. $a^3+b^3+c^3-3abc$. **35.** $x^3-y^3-z^3-3xyz$.

36. $y^3+x^3-1+3xy$. **37.** $x^3y-x^3z+y^3z-y^3x+z^3x-z^3y$.

38. $x^5+4x^3-6x^4-28x^2+1$. **39.** $-1\frac{1}{2}a^6+2\frac{1}{4}a^5b+1\frac{6}{16}\frac{1}{4}a^2b^2+\frac{1}{16}ab^5-\frac{1}{32}b^6$

40. $a^6-6a^5x+15a^4x^2-20a^3x^3+15a^2x^4-6ax^5+x^6$.

41. $-3x^9+14x^8+x^7-10x^6+6x^5+4x^4-6x^3+x^2+4x-1$.

42. $2a^{11}-a^9-a^8+6a^7-10a^6-2a^5+8a^4-5a^3-2a^2+5a-1$.

43. $1-a^4$. **44.** $a^3+6a^2+11a+6$. **45.** a^3+6a^2-a-30.

46. $x^3+x^2(a+b+c)+x(ab+ac+bc)+abc$.

47. $x^3+6x^2y+11xy^2+6y^3$. **48.** $a^3+3a^2b+3ab^2+b^3$.

49. $a^3-3a^2b+3ab^2-b^3$. **50.** x^6-y^6. **51.** $a^6+a^3b^3+b^6$.

52. $a^{12}-b^{12}$. **53.** $2a^2b^2+2b^2c^2+2c^2a^2-a^4-b^4-c^4$.

54. $a^2+b^2+c^2+2ab+2ac+2bc$. **55.** $a^2+b^2+c^2-2ab+2ac-2bc$.

56. $4a^2+9b^2+16c^2-12ab+16ac-24bc$.

57. $a^3+b^3+c^3+3a^2b+3a^2c+3b^2a+3b^2c+3c^2a+3c^2b+6abc$.

58. $a^3-b^3+c^3-3a^2b+3a^2c+3b^2c-3bc^2+3ab^2+3ac^2-6abc$.

59. -1. **60.** 2. **61.** -4. **62.** $-3y^2$. **63.** -1. **64.** 39.

65. -23. **66.** $-2x^2$. **67.** $-47x$. **68.** $-3x$. **69.** $-3y$.

70. $12x^2-4xy+4y^2$. **71.** $3x^2-3y^2$. **72.** $4x^4+8x^2a^2+4a^4$.

73. $-14x^2-5x+21$. **74.** $\frac{x^2+79x+66}{12}$. **75.** $9x^3-12x^2-9x+11$.

76. $2x^3+4x$. **77.** $30x^4+89x^3-26x^2-28x+13$.

78. $x^{2m+2n}+x^{m+n}y^{m-n}-2y^{2m-2n}$.

79. $6x^4+11x^3+10x^2+7x+2,72072$.

80. $ar-bq+pc$. **81.** (i) $a^3+b^3+c^3-3abc$, (ii) 0, (iii) 0.

EXERCISE 15. (Page 64.)

13. $bd-xm$. **14.** $\frac{1}{2}by-\frac{1}{3}cz+\frac{1}{6}bcxy$. **15.** $ab+bc+ca$.

16. $5a^2x^3-4ax^2-7x$. **17.** $-3a^2b^2+2x-abx^2$.

18. $-2a^3+3ab^2+5c^3$. **19.** $40c^2-45b^2+24a^2$.

20. $6bc^2-8a^2c+12ab^2$. **21.** $x+3$. **22.** $x+5$. **23.** $x-3$.

$x-7$. 25 $x-2$. 26. $5x+3$. 27. $3m-2$. 28. $a+3$.

$2a-3$. 30. $2p+1$. 31. $m-3$. 32. $3a+2$. 33. $1-x$.

a^2-2a+1 35. $x^2-3xy+y^2$. 36. a^2-ab-b^2.

$2a^2-a+3$. 38. a^2+3a+2. 39. $4a^2+3a-2$. 40. a^2-2a+3.

$-2x^2+3x-1$. 42. a^2-5a+1. 43. $a+1$. 44. $p^2-2pq+3q^2$.

x^2+x+1. 46. x^2-xy+y^2. 47. $x^4-x^3y+x^2y^2-xy^3+y^4$.

$x^4+x^3y+x^2y^2+xy^3+y^4$.

$x^7-x^6y+x^5y^2-x^4y^3+x^3y^4-x^2y^5+xy^6-y^7$.

$x^7-x^6+x^4-x^3+x-1$. 51. $a^2+b^2+c^2-ab-ac-bc$.

x^3-2x^2+2x-1. 53. $x^6+2x^3+3x^2+2x+1$.

$x^6-x^3y^3+y^6$. 55. $x^2+y^2-xy+x+y+1$.

$x^3-2x^2y+2xy^2-y^3$. 57. $x^8+x^6y^2-3x^2y^6-3y^8$.

$a^3+\frac{1}{3}a^2b+\frac{1}{9}ab^2+\frac{1}{27}b^3$. 59. $x^2-\frac{3}{4}x+1$. 60. $\frac{1}{2}x+\frac{1}{3}y-\frac{1}{5}z$.

$x^2+y^2+z^2+1$. 62. $a-b-c$. 63. $x^2+4y^2+9z^2-2xy+3xz+6yz$.

$a+2b+3c$. 65. $x^4+x^3y+x^2y^2+xy^3+y^4$.

$x^{2n}+x^ny^n+y^{2n}$. 67. $a+b-1+\dfrac{4}{1+a+b}$.

$a+2+\dfrac{5(a+1)}{a^2+5a+6}$. 69. $a^2+4a+1+\dfrac{2a+3}{a^2+3a+1}$.

$2a^2-ab+b^2-\dfrac{3b^4}{a^2+ab-2b^2}$.

Remainder $=x^4$, complete quotient $=1-x+x^2-x^3+\dfrac{x^4}{1+x}$.

Remainder $=135a^4$, complete quotient $=1-5a+15a^2-45a^3+\dfrac{135a^4}{1+3a}$.

Remainder $=-4a^5+8b^6$, complete quotient $=1+a-2a^3-4a^4+\dfrac{-4a^5+8a^6}{1-2a+2a^2}$.

$x^4-4x^3+6x^2-4x+1$. 79. x^2-3x+2. 80. -2.

-2. 82. $5a+4$. 83. $m=-4$. 84. -36. 85. $\dfrac{5}{m}$.

$a=-9$, $b=30$.

EXERCISE 17. (Page 76.)

1. 6. 2. 7 3. 3. 4. 4. 5. 28. 6. 18.
7. 40. 8. 48. 9. 18. 10. 10. 11. 18. 12. 15.
13. 13. 14. 17. 15. 13. 16. 11. 17. 7. 18. 9.
19. 9. 20. 11. 21. 6. 22. 6. 23. 5. 24. 4.
25. 3. 26. 3. 27. 6. 28. 9. 29. 9. 30. 7.
31. 1. 32. $\frac{12}{5}$. 33. 3. 34. $\frac{5}{2}$. 35 $\frac{10}{3}$. 36. $\frac{10}{3}$.
37. 6. 38. 6. 39. 4. 40. 3. 41. 9. 42. $\frac{3}{2}$.
43. $1\frac{5}{9}$. 44. $2\frac{1}{10}$ 45. $1\frac{2}{3}$. 46. 3. 47. 13. 48. 10.
49. 2. 50. 3. 51. 3. 52. $\frac{2}{3}$. 53. $2\frac{1}{3}$. 54. $4\frac{3}{7}$.
55. 2. 56. -3. 57. -11. 58. $-\frac{4}{3}$. 59. -4. 60. 1.
61. -7. 62. $-\frac{1}{2}$. 63. 2. 64. $2\frac{5}{6}$. 65. $\frac{3}{2}$. 66. $-\frac{6}{5}$.
67. $-\frac{10}{19}$. 68. 5. 69. 0. 70. $4\frac{2}{3}$. 71. $2\frac{4}{9}$. 72. 1. 73 2.
74. $2\frac{1}{4}$. 75. $\frac{2}{3}$. 76. 7. 77. -4. 78. $-3\frac{3}{11}$. 79. $1\frac{2}{13}$.
80. $4\frac{2}{3}$. 81. $-5\frac{7}{3}$. 82. $\frac{17}{15}$. 83. -3. 84. -2. 85. $-\frac{1}{3}$.
86. $\frac{2}{7}$. 87. $\frac{5}{19}$. 88. $-\frac{1}{19}$. 89. $2\frac{3}{7}$. 90. 10. 91. 1. 92. $8\frac{1}{3}$.
93. -1. 94. 1. 95. $-3\frac{1}{10}$. 96. $3\frac{16}{25}$. 97. 14. 98. $-\frac{1}{2}$.
99. $1\frac{2}{5}$. 100. (i) -1, (ii) any value of x, (iii) no value of x.

EXERCISE 18. (Page 83.)

1. 6. 2. 36. 3. 3. 4. 24. 5. 9. 6. $\frac{5}{6}$.
7. $\frac{3}{2}$ 8. $3\frac{3}{7}$. 9. $-\frac{16}{23}$ 10. $4\frac{1}{2}$ 11. 3. 12. $17\frac{1}{7}$. 13. 10.
14. -11. 15. 11. 16. 24. 17. 24. 18. $54\frac{8}{9}$. 19. $\frac{1}{2}$.
20. 2. 21. $3\frac{1}{2}$. 22. $1\frac{2}{3}$. 23. $-1\frac{31}{85}$. 24. $3\frac{3}{19}$. 25. 1.
26. $-\frac{5}{24}$. 27. 10. 28. $-\frac{28}{37}$. 29. $1\frac{5}{7}$. 30. $2\frac{1}{5}$. 31. $1\frac{3}{4}\frac{3}{5}$.
32. $\frac{15}{154}$. 33. $-\frac{185}{73}$. 34. $-1{\cdot}67$. 35. 24.

EXERCISE 19. (Page 86.)

1. (i) $b=\frac{2A}{h}$, (ii) $h=\frac{2A}{b}$. 2. $r=\frac{C}{2\pi}$. 3. (i) $\frac{2A}{a+b}$,
(ii) $(a+b)=\frac{2A}{h}$. 4. (i) $r=\frac{A}{2\pi h}$, (ii) $h=\frac{A}{2r\pi}$.
5. $r=\sqrt{\frac{A}{\pi}}$. 6. $c=\sqrt{\frac{A}{6}}$. 7. (i) $x=180-2y$,
(ii) $y=\frac{180-x}{2}$. 8. (i) $r=100\left(\frac{y}{x}-1\right)$, (ii) 5.

9. (i) $r=\sqrt{\frac{V}{\pi h}}$. (ii) $h=\frac{V}{r^2\pi}$. 10. (i) $f=\frac{v-u}{t}$, (ii) $t=\frac{v-u}{f}$.

11. $t=\frac{\sqrt{S}}{4}$. 12. $a=\sqrt{c^2-b^2}$. 13. (i) $C=\frac{5}{9}(F-32)$, (ii) 10.

14. $h=\frac{3V}{\pi r^2}$, $r=\sqrt{\frac{3V}{\pi h}}$. 15. $f=\frac{2(s-ut)}{t^2}$.

16. $t=\frac{pv-2730}{10}$, $t=42$.

EXERCISE 21. (Page 94.)

1. 75°. 2. 102°. 3. 36°, 72°, 72°. 4. 25°, 5°.
5. 48 years, 24 years. 6. 45 years, 15 years. 7. 69°, 37°.
8. 50, 33. 9. 10. 10. 19. 11. 30. 12. 170.
13. 28. 14. 24. 15. 25. 16. Rs. 43, Rs. 53.
17. Rs. 81, Rs. 69. 18. 16, 21, 12. 19. Rs. 20, Rs. 27, Rs. 34.
20. Rs. 90, Rs. 120, Rs. 130. 21. Rs. 200, Rs. 160, Rs. 140.
22. 10. 23. 60. 24. 420. 25 60. 26. 36, 45.
27. $20\frac{4}{7}$ ft. 28. 129 miles 29. 56 gallons. 30. 7200.
31. 84. 32. 360 33. 87. 34 420, 440. 35. 12, 28.
36. 566, 112. 37. 854, 210. 38. 645, 214. 39. 9, 24.
40. 16, 36. 41. 21, 45. 42. 48. 43. 56, 28. 44. 54, 44.
45. 34, 8. 46. 28, 29, 30. 47. 17, 18, 19, 20. 48. 14, 16, 18.
49. 25, 27, 29. 50. 50, 51. 51. 110. 112. 52. 75, 77.
53. 102, 104 54. 115, 117. 55. 48. 56. 25. 57. 71.
58. 53. 59. 62. 60. 21. 61. 72. 62. 24.
63. 39. 64. $27\frac{3}{7}$ minutes past 8 P.M. 65. $12\frac{1}{2}$ days after the start of the first steamer, 3000 miles from P.
66. $1\frac{10}{11}$ hours. 67. 3 miles. 68. 20 miles an hour.
69. $5\frac{5}{6}$ miles. $21\frac{2}{3}$ miles. 70. 48 miles. 71. A meets B after travelling
72. 30 miles per hour; 240 miles.
73. After $10\frac{10}{11}$ minutes; B overtakes A after travelling $10\frac{10}{11}$ miles.
74. (i) After $34\frac{2}{11}$ minutes, (ii) after $27\frac{3}{11}$ minutes,
(iii) after $43\frac{7}{11}$ minutes. 75. (i) $10\frac{10}{11}$ minutes past 2,
(ii) $21\frac{7}{11}$ minutes past 4. 76. (i) $54\frac{6}{11}$ minutes past 4,
(ii) $43\frac{7}{11}$ minutes past 2. 77. (i) $43\frac{7}{11}$ minutes past 5, (ii) $49\frac{1}{11}$ minutes past 6. 78. 32 yds. and 48 yds. 79. Tea 1/4 as. per lb.

and coffee 10 as. per lb. 80. 126 mangoes. 81. 20 seers of the first kind, 30 seers of the second kind. 82. Rs. 5400.
83. Rs. 1200, Rs. 2000. 84. Rs. 2400. 85. 3400 males, 3200 females. 86. 40 goats of each kind. 87. 44 men.
88. 255 ft. 89. 49 ft., 40 ft. 90. 16 ft., 10 ft.
91. 27 ft., 15 ft. 92. Rs. 170, 310 eight-anna bits.
93. 60 four-anna bits, 20 two-anna bits. 94. 20, 30, 5, 125.
95. 40, 31, 72, 12. 96. 19, 13, 7, 42. 97. 28 boys, Rs. 125.
98. 5 hours, 17 miles. 99. 160 leaps. 100. 40 leaps.

SECTIONAL REVISION I. (Page 110.)

Paper 1. 2. $-35, -11, -5$. 3. $x^2+xy-xz-yz$.
5. $-\frac{1}{5}$. 6. 48.

Paper 2. 1. $a^2+b^2+c^2$. 2. 3. 3. $\frac{6a-5b}{3}$.
4. a^6-b^6. 5. $-4\frac{2}{5}$. 6. 15 years, 35 years.

Paper 3. 1. $6a-2b$. 3. Quotient $=1+3x+5x^2+7x^3$,
remainder $=-4x^4-4x^5$. 4. $20x^2-5xb$. 5. $\frac{1}{5}$.
6. $3\frac{5}{9}$ hours after the departure of B; $16\frac{8}{9}$ miles from the starting place.

Paper 4. 1. $-a+b+4c$. 2. x^2+3x-1. 3. $x^{3n}-1$.
4. 1. 6 $\frac{5}{12}$.

Paper 5. 2. (i) +1, (ii) +1, (iii) +1, (iv) −1. 3. Quotient $=1-a+a^3-a^4$, remainder $=a^6$. 4. 5. 5. a^8+a^4+1.
6. $46\frac{4}{3}$ miles.

Paper 6. 1. $2(ab+ac+2bc)$. 2. $-56+68x-158x^2+107x^3-98x^4+47x^5-15x^6+x^7$.
3. $11am^2$. 4. x^2.
5. 2. 6. 48 miles.

Paper 7. 1. (i) 20. (ii) $9a^2+3a-5$. 2. $3a^2+8b^2-22c^2-16ab-86ac+86bc$ or $22c^2-8a^2-8b^2+15ab+86ac-86bc$.
3. $18x^2+40xy$. 4. $10a^2+2$. 5. -1. 6. Rs. 2300.

Paper 8. 2. $2p, 2q$. 3. $a^4-2a^3+3a^2+5a+11$.
5. $-7\frac{13}{17}$. 6. 24.

Paper 9. 1. $11+7x-10x^2+x^3+2x^4-4x^5$. 2. 25, 121, 361, 841...

3. (i) $r=\sqrt{\frac{3v}{\pi h}}$. (ii) $h=\frac{3v}{\pi r^2}$. 4. $13x^2$. 5. $-\frac{5}{6}$.

6. 31 ft., 20 ft.

Paper 10. 1. (i) $\frac{2}{3}a^4b^6c^8d^2$. (ii) $\frac{4}{5}a^2b^5cd^2$.

2. $\frac{mn}{100}\%$ 3. (i) 30, (ii) 1. 4. $3x^3-2x^2-3x-6$.

5. (i) $3x-2y-2z$, (ii) $k=-a$. 6. 280 leaps.

EXERCISE 22. (Page 116.)

1. $4x^2+20xy+25y^2$. 2. $9a^2+42ab+49b^2$. 3. $25x^2+30x+9$.
4. $9+24b+16b^2$. 5. $9a^2b^2+24ab^3+16b^4$.
6. $64a^4+48a^3b+9a^2b^2$. 7. $81a^2x^2+72axy^2+16y^4$.
8. $m^4+2m^2n^2+n^4$. 9. $p^2x^2+2pqxy+q^2y^2$.
10. $p^6+2p^4q^2+p^2q^4$. 11. $4a^4b^2+12a^3b^3+9a^2b^4$.
12. $a^2-2ab+b^2$. 13. $56xy$. 14. $25n^2$. 15. $49q^2$.
16. $108x$. 17. 9. 18. $16p^2$ 19. 91204. 20. 164025. 21. 494209.
22. 649636. 23. 373321. 24. 1002001. 25. $5a^2+8ab+5b^2$.
26. $5x^2-5y^2$. 27. $p^2x^2+q^2y^2-q^2x^2-p^2y^2$.
28. $n^2+2mnx-2mxk-k^2$. 29. $9a^2-30ab+25b^2$.
30. $25m^2-30m+9$. 31. $36p^2-60p+25$.
32. $49x^2-14x+1$. 33. $16-56p+49p^2$.
34. $a^2+2ab+b^2$. 35. $16m^4-40m^2n^2+25n^4$.
36. $a^2x^2-2abxy+b^2y^2$. 37. $4a^4x^2-12a^2b^2xy+9b^4y^2$.
38. $49a^4-56a^2b^2+16b^4$. 39. $4a^6b^2-4a^3b^4+b^6$.
40. $49p^6q^4-112p^5q^5+64p^4q^6$. 41. $24ab$. 42. $16y^2$.
43. $16q^2$. 44. $154m$. 45. $36q^2$. 46. $25x^2$.
47. 39204. 48. 60025. 49. 156816.
50. 19044. 51. 247009. 52. 358801.
53. $4ab$. 54. $13a^2-24ab+13b^2$. 55. $4pq.xy$.
56. $3m^2x^2-10mnxy-8n^2y^2$. 57. $17a^2b^2-12ab^3-4a^4$.
58. 369. 59. 185. 60. 289. 61. 305.

EXERCISE 23. (Page 119.)

1. $9x^2-25$. 2. $a^2m^2-b^2n^2$. 3. $a^2x^2-b^4$.
4. $p^2x^2-q^2y^2$. 5. $-16a^2-25b^2$. 6. $9x^2y^2-49x^2z^2$.
7. a^4-b^4. 8. a^4-1. 9. a^8-b^8.
10. $x^2-y^2+2yz-z^2$. 11. $x^2-6xy+9y^2-16z^2$.
12. $x^4+x^2y^2+y^4$. 13. x^4+x^2+1. 14. x^8+x^4+1.

15. $3a^3-6ab+8ac+2bc-3c^2$. 16. $4a^3b+4ab^3$
17. $4ab-4ac+4ad$. 18. $24ab-32ac+40ad$. 19. 5445.
20. 7815. 21. 6460. 22. 8547.

EXERCISE 24. (Page 120.)

25. $4x^2+16x+15$. 26. $9m^2+9m-10$.
27. $1-x-6x^2$. 28. $9-3x-20x^2$.
29. $x^2-2xy+y^2+3zx-3zy+2z^2$.
30. $4x^2+4xy+y^2+4zx+2yz-3z^2$.
31. $4x^2+12xy+9y^2-8xz-12yz-5z^2$.

EXERCISE 25. (Page 122.)

1. $x^3+6x^2+11x+6$. 2. $x^3+9x^2+26x+24$.
3. $x^3+15x^2+74x+120$. 4. $x^3+2x^2-23x-60$.
5. $x^3+2x^2-43x-140$. 6. $x^3+3x^2-18x-40$.
7. $x^3-6x^2-7x+60$. 8. $x^3-5x^2-18x+72$.
9. $x^3-12x^2+47x-60$. 10. $x^3-12x^2+39x-28$.
11. $x^3-10x^2+31x-30$. 12. $x^3-18x^2+104x-192$.
13. $8x^3+36x^2+46x+15$. 14. $27x^3-63x^2+42x-8$.
15. $a^3x^3+11a^2x^2+36ax+36$. 16. $m^3x^3-8m^2x^2+11mx+20$.
17. $p^3x^3-14p^2x^2+61px-84$.
18. $(x+p)(x+q)(x+r)=x^3+(p+q+r)x^2+(pq+pr+qr)x+pqr$.
19. $(p+a)(p+2b)(p-3c)=(p)^3+(a+2b-3c)p^2+(2ab-3ac-6bc)p$ $+(-6abc)$. 20. $(m+3)(m-4)(m+5)=m^3+(3-4+5)m^2$ $+\{3\times(-4)+3\times5+(-4)5\times\}m-60$.

EXERCISE 26. (Page 123.)

1. $8x^3+12x^2y+6xy^2+y^3$. 2. $x^3+9x^2y+27xy^2+27y^3$.
3. $8a^3+36a^2b+54ab^2+27b^3$. 4. $a^3x^3+3a^2x^2by+3axb^2y^2+b^3y^3$.
5. $1+9x+27x^2+27x^3$. 6. $8+60x+150x^2+125x^3$.
7. $27x^6+54x^4+36x^2+8$. 8 $a^6+3a^4b^2+3b^4a^2+b^6$.
9. $27a^6+54a^4b^2+36a^2b^4+8b^6$.
10. $(4m+2n)^3=(4m)^3+3(4m)^2(2n)+3(4m)(2n)^2+(2n)^3$.
11. $(2p+3q)^3=(2p)^3+3(2p)^2(3q)+3(2p)(3q)^2+(3q)^3$.
12. $(5p+2q)^3=(5p)^3+3(5p)^2(2q)+3(5p)(2q)^2+(2q)^3$.
13. $(2a+3b)^3=(2a)^3+(3b)^3+3(6ab)(2a+3b)$.
14. $343a^3+294a^2b+84b^2a+8b^3$. 15. $125(x^3+3x^2y+3xy^2+y^3)$.
16. $2a^3+9a^2+15a+9$. 17. $24a^2+48a+26$.
18. 35. 9. 91. 20. 243. 21. 355. 22. -19. 23. -105.

26. $8x^3-12x^2+6x-1$. **27.** $27-54x+36x^2-8x^3$.
28. $27a^3-108a^2b+144ab^2-64b^3$. **29.** $64x^3-144x^2y+108xy^2-27y^3$.
30. $125x^3-300x^2y+240xy^2-64y^3$.
31. $64m^6-144m^4n+108m^2n^2-27n^3$.
32. $8a^3x^3-36a^2x^2by+54b^2y^2ax-27b^3y^3$.
33. $(2x-3y)^3=(2x)^3-3(2x)^2(3y)+3(2x)(3y)^2-(3y)^3$.
34. $(4m-5x)^3=(4m)^3-3(4m)^2(5x)+3(4m)(5x)^2-(5x)^3$.
35. $(4a-6b)^3=(4a)^3-3(4a)^2(6b)+3(4a)(6b)^2-(6b)^3$.
36. $(2p-3q)^3=(2p)^3-(3q)^3-3(2p)(3q)(2p-3q)$.
37. $27(m^3-12m^2n+48mn^2-64n^3)$. **38** $p^3+15p^2q+75pq^2+125q^3$.
39. $12x^2+24x+28$. **40.** $96x^2-96x+152$.
41. 189. **42.** 335 **43.** 448. **44.** -3. **45.** -7. **46.** 988.

EXERCISE 27. (Page 126.)

1. x^3+27. **2.** x^3+125. **3.** x^3+343. **4.** x^3+1331.
5. $125a^3+27$. **6.** $27m^3+125n^3$. **7.** $x^3y^3+64z^3$. **8.** $64x^3+729y^3$.
9. $a^6+b^3c^3$. **10.** a^6+64b^6. **11.** x^3+1. **12.** 98.
13. $9(x^3+y^3)$. **14.** $37(x^3-y^3)$. **15.** 56. **16.** x^3-8.
17. x^3-216. **18.** x^3-512. **19.** x^3-1728.
20. $27x^3-8$. **21.** $125x^3-27y^3$. **22.** $p^3q^3-8r^3$.
23. $a^6-b^3c^3$. **24.** $19m^3-37n^3$. **25.** $37(p^3+q^3)$.
26. -189. **27.** $(a)^3+(3b)^3=(a+3b)\{(a)^2-(a)(3b)+(3b)^2\}$.
28. $(2x)^3-(5y)^3=(2x-5y)\{(2x)^2+(2x)(5y)+(5y)^2\}$.
29. $(4p)^3+(3q)^3=(4p+3q)\{(4p)^2-(4p)(3q)+(3q)^2\}$.
30. $(7a)^3-(6b)^3=(7a-6b)\{49a^2+(7a)(6b)+36b^2\}$.

EXERCISE 28. (Page 128.)

1. $a^2+b^2+c^2+2ab-2ac-2bc$. **2.** $a^2+b^2+c^2-2ab-2ac+2bc$.
3. $4a^2+b^2+c^2-4ab-4ac+2bc$. **4.** $4a^2+b^2+c^2+4ab-4ac-2bc$.
5. $4a^2+b^2+9c^2-4ab+12ac-6bc$.
6. $25x^2+4y^2+9z^2-20xy+30xz-12yz$.
7. $4x^2+9y^2+1-12xy+4x-6y$.
8. $x^4+3x^2y^2+y^4-2x^3y-2xy^3$.
9. $p^2x^4+q^2x^2y^2+r^2y^4-2pqx^3y+2prx^2y^2-2qrxy^3$.
10. $4a^2+b^2+9c^2+d^2+4ab+12ac+4ad+6bc+2bd+6cd$.
11. $a^2+4b^2+9c^2+16d^2-4ab-6ac+8ad+12bc-16bd-24cd$.
12. $4p^2+9q^2+r^2+4s^2+t^2-12pq+4pr-8ps+4pt-6qr+12qs$
$-6qt-4rs+2rt-4st$.
13. $9x^2+16y^2+4z^2+w^2-24xy+12xz-6xw-16yz+8yw-4zw$.
14. $3a^2-6ab+8ac+2bc-3c^2$. **15.** $2a^2+4ab+8ac+2bc-b^2-c^2$.

16. $4a^2+4b^2-2c^2+10ab-6ac+6bc$.
17. $2(a^2+b^2+c^2+d^2+2ac+2bd)$.
18. $2(a^2+b^2+c^2+d^2-2ac-2ad+2cd)$.
19. $2a^2-b^2-c^2-d^2-4ab+4ac-4ad+2bc-2bd+2cd$.
20. 361. 21. 841. 22. 20. 23. 52.

EXERCISE 29. (Page 130.)

1. 169. 2. 324. 3. 4. 4. 81. 5. 28. 6. 105.
7. 3. 8. ·61. 9. 3. 10. 3. 11. 12. 12. 23,527.
13. 11, 119. 14. p^4+4p^2+2. 15. 110. 16. 14. 17. m^3-3m.
18. a^3+3a. 19. $(3x+3y-z)^2$. 20. $(2x+5y+z)^2$.
21. $(x-y+z)^2$. 22. $(x-2y+3z)^2$. 25. 124. 26. 35,984.

EXERCISE 30. (Page 134.)

1. $x^3+y^3-1+3xy$.
2. $x^3-y^3-1-3xy$.
3. $8x^3+y^3+1-6xy$.
4. $x^3+8y^3-z^3+6xyz$.
5. $x^3+8y^3-27z^3+18xyz$.
6. $8x^3-y^3-27z^3-18xyz$.
7. $9x^2+y^2+z^2+3xy+3xz-yz$.
8. $5x+3y-z$.
9. -572.
10. 63.
11. 740.

EXERCISE 31. (Page 136.)

1. $4a^2+2ax+x^2$.
2. $a^2-4ab+16b^2$.
3. $32a^5-16a^4x+8a^3x^2-4a^2x^3+2ax^4-x^5$.
4. $9m^2+3m+1$.
5. $x^3+4x^2y+16xy^2+64y^3$.
6. $1-2m+4m^2-8m^3$.
7. $16a^4-8a^3b+4a^2b^2-2ab^3+b^4$.
8. $m^{15}-m^{12}n^2+m^9n^4-m^6n^6+m^3n^8-n^{10}$.
9. $\frac{1}{9}a^2-\frac{1}{3}ab+b^2$.
10. $x^6-x^4yz+x^2y^2z^2-y^3z^3$.
11. $p+q-r$.
12. $a^2+b^2+c^2+ab-ac-2bc$,
13. $m^{15}+m^{12}n^2+m^9n^4+m^6n^6+m^3n^8+n^{10}$.
14. $x^8-x^7a+x^6a^2-x^5a^3+x^4a^4-x^3a^5+x^2a^6-xa^7+a^8$.
15. $x^7+x^6a+x^5a^2+x^4a^3+x^3a^4+x^2a^5+xa^6+a^7$.

EXERCISE 32. (Page 138.)

21. $(a+b)(c+d)$.
22. $(p+q)(p-q)$.
23. $(x-1)(7m+2)$.
24. $(x^2+x+1)(2a+3b)$.
25. $(ax+b)(3p+q+2r)$.
26. $(pq+r)(m^2+mn+n^2)$.
27. $(x-y)(a-b+c+d)$.
28. $x(x+7)(x^2+7x-25)$.
29. $5(p^2-qr)^2(3p^2-3qr-2)$.
30. $2(x-y)(3p+q)$.
31. $(x+y)\{(x+y)^2-3xy\}$.

EXERCISE 33. (Page 140.)

1. $(x+y)(a+5)$.
2. $(x-y)(a-b)$.
3. $(x-y)(m+3)$.
4. $(x+2)(y+3)$.
5. $(x+b)(x-3a)$.
6. $(a+3)(a^2+9)$.
7. $(3p-5)(p^2+2)$.
8. $(2p-3q)(3p+2r)$.
9. $(p+1)(q+1)$.
10. $(a+1)^2(a-1)$.
11. $(2a-5)(a^2+1)$.
12. $(5a-2)(a^2+1)$.
13. $(x-y)(x-3)$.
14. $(2a+b)(3a-c)$.
15. $(2a-1)(a^3+2)$.
16. $(y-1)(y^2-a+1)$.
17. $(11x^2+7)(x+5)$.
18. $(x^2+1)(ax+b)$.
19. $(3y-2)(2x-3)$.
20. $(x-2a)(x-b)$.
21. $(x+1)(ax-1)$.
22. $(ax-1)(bx+y)$.
23. $(a+1)(a^2x-ay-z)$.
24. $(a+bx+cx^2)(x+1)(x-1)$.
25. $(kx^2+lx+m)(x+1)(x-1)$.
26. $(a-2b)(a-2b+3)$.
27. $(a+bc)(b+ca)$.
28. $(ax-by)(bx-ay)$.

EXERCISE 34. (Page 141.)

1. $(2a+7)^2$.
2. $(5p-3)^2$.
3. $(5a+4b)^2$.
4. $(7m-4n)^2$.
5. $(6a+5b)^2$.
6. $(6a-7b)^2$.
7. $(7a+3b)^2$.
8. $(5x-6y)^2$.
9. $(4x+7y)^2$.
10. $(9a-7b)^2$.
11. $(9x+5y)^2$.
12. $(3m-11n)^2$
13. $(8x+y)^2$.
14. $(4p-6q)^2$.
15. $(3x+9y)^2$.
16. $(5m-7n)^2$.
17. $\left(x+\frac{1}{x}\right)^2$.
18. $\left(\frac{x}{y}-\frac{y}{x}\right)^2$
19. $(ax+3by)^2$.
20. $(4ab-c^2)^2$.
21. $(2a+3b+c+d)^2$.
22. $(3p-q-r-s)^2$.

EXERCISE 35. (Page 142)

1. $(4x+3)(4x-3)$.
2. $(9a+5b)(9a-5b)$.
3. $(1+6x)(1-6x)$.
4. $(8a+1)(8a-1)$.
5. $(3x+10)(3x-10)$.
6. $(7+5a)(7-5a)$.
7. $(11+c)(11-c)$.
8. $(1+7ac)(1-7ac)$.
9. $(a^2+5)(a^2-5)$.
10. $(4x^3+7)(4x^3-7)$.
11. $(1+9x^2y)(1-9x^2y)$.
12. $(x^3+6y)(x^3-6y)$.
13. $(9x^2+1)(3x+1)(3x-1)$.
14. $x(4x^2+9)(2x+3)(2x-3)$.
15. $(1+4x^2)(1+2x)(1-2x)$.
16. $a^2(1+9a^2)(1+3a)(1-3a)$.
17. $x^3(6x^2+5a^2)(6x^2-5a^2)$.
18. $3x^5(8x^2+9y^2)(8x^2-9y^2)$.
19. $4a^5b^3(11a^6b^3+9)(11a^6b^3-9)$.
20. $(2x-3y)(2x-3y+2)(2x-3y-2)$.
21. $(a+5b+7c)(a+5b-7c)$.
22. $(x+2y-3z)(x-2y+3z)$.
23. $4ab$.
24. $(5x-y)(x-3y)$.
25. $(7x-y)(x-7y)$.
26. $(13a+b)(a+13b)$
27. $(19m+13n)(23n-m)$.
28. $5n(12m-13n)$.
29. $12(5a-1)(a+2)$.
30. $4a(b-c)$.
31. $(10a+9b-7c)(15b-2a-25c)$.
32. $28a(3a-5)$.
33. $3ax(x^2+a^2)$.
34. 23935.
35. 714000.
36. ·476.

EXERCISE 36. (Page 143.)

13. $(a^2-2b^2)(a^4+2a^2b^2+4b^4)$.
14. $(3m^2+5n^2)(9m^4-15m^2n^2+25n^4)$.
15. $(a-b)(a^2+ab+b^2)(a^6+a^3b^3+b^6)$.
16. $(a^3-2b^2c^2)(a^6+2a^3b^2c^2+4b^4c^4)$.
17. $ab(7a^3+4b^4)(49a^8-28a^4b^4+16b^8)$.
18. $(3x+2y-z)(9x^2+4y^2+z^2+12xy+3xz+2yz)$.
19. $(2a-3b+3c)(4a^2+9b^2+9c^2-12ab-6ac+9bc)$.
20. $(a-b)(7a^2+13ab+7b^2)$.
21. $(a-2b+1)(a^2-4ab+4b^2-a+2b+1)$.
22. $(x-y-1)(x^2-2xy+y^2+x-y+1)$.
23. $2(b-2a)(151a^2-466ab+364b^2)$.
24. $(18x-5y)(156x^2+240xy+277y^2)$. 25. 2240. 26. 15500.
27. 31040. 28. $(a-b)(a+b)(a^2+ab+b^2)(a^2-ab+b^2)$.
29. $(a-b)(a+b)(a^2+b^2)(a^2+ab+b^2)(a^2-ab+b^2)(a^4-a^2b^2+b^4)$.
30. $(2a^2+1)(2a^2-1)(4a^4-2a^2+1)(4a^4+2a^2+1)$.
31. $(3a+b)(3a-b)(9a^2=3ab+b^2)(9a^2+3ab+b^2)$.
32. $a(a-b)(a+b)(a^2+ab+b^2)(a^2-ab+b^2)$.
33. $3x^7(x+2y)(x-2y)(x^2-2xy+4y^2)(x^2+2xy+4y^2)$.

EXERCISE 37. (Page 146.)

1. $(x+6)(x+7)$.
2. $(a-13)(a-5)$.
3. $(x+13)(x+5)$.
4. $(p-9)(p-6)$.
5. $(x+12)(x+6)$.
6. $(x+13)(x+7)$.
7. $(x-13)(x-9)$.
8. $(x+8)(x+13)$.
9. $(a+13)(a-8)$.
10. $(x+9)(x+12)$.
11. $(a+12)(a-9)$.
12. $(x+12)(x+15)$.
13. $(p+13)(p-12)$.
14. $(a-15)(a+12)$.
15. $(a-13)(a+11)$.
16. $(a-16)(a+15)$.
17. $(x+15)(x-7)$.
18. $(a-30)(a+4)$.
19. $(1-15x)(1-4x)$.
20. $(1-51a)(1+2a)$.
41. $(p-10q)(p+2q)$.
42. $(m-8n)(m-4n)$.
43. $(a+6b)(a-5b)$.
44. $(a-9b)(a+5b)$.
45. $(p-16q)(p+3q)$.
46. $(a+8b)(a+12b)$.
47. $(m+7n)(m-4n)$.
48. $(x-16y)(x+5y)$.
49. $(a-13b)(a+7b)$.
50. $(x-15y)(x+9y)$.
51. $(a-2)(a+2)(a^2+3)$.
52. $(x^2+7)(x+2)(x-2)$
53. $(x^2+7)(x+5)(x-5)$
54. $(x^2-17)(x^2-8)$.
55. $(x^2+7y^2)(x+3y)(x-3y)$.
56. $(a-1)(a^2+a+1)(a^3+3)$.
57. $(p-1)(p+3)(p^2+p+1)(p^2-3p+9)$.
58. $(x^3-12)(x^3-15)$.
59. $(m-2n)(m+2n)(m^2+4n^2)(m^4+5n^4)$.
60. $(p^2+q^2)(p^2-2q^2)(p^4-p^2q^2+q^4)(p^4+2p^2q^2+4q^4)$.
61. $(a^2-2a-6)(a^2-2a+5)$.
62. $(x^2+x-7)(x^2+x+5)$.
63. $(m^2-3m-6)(m^2-3m-2)$.
64. $(p^2-5p+9)(p^2-5p-4)$.

ANSWERS

65. $(a-2)^2(a^2-4a+12)$.
66. $(x^2-6x-15)(x^2-6x+12)$.
67. $(m+3)(m+4)(m^2+7m-13)$.
68. $(p^2+4p+14)(p^2+4p+7)$.
69. $(x-3)(x-5)(x^2-8x-16)$.
70. $(m^2-7m-15)(m^2-7m+13)$.
71. $2(2x-y)(5x-3y)$.
72. $(7y-4x)(4x-y)$.
73. $(6a+19b)(a+9b)$.
74. $-2(5m+3n)(m+5n)$.
75. $10a(7b-a)$.
76. $-(3m+17n)(9m+7n)$.
77. $(3a-8b)(13b-4a)$.
78. $(a-4b)(a-b)^2(a+2b)$.
79. $7b^2(9a^2-11b^2)$.
80. $(a-6b)(a-4b)^2(a-3b)$.
81. $\frac{x+2}{x}$.
82. $\frac{x+1}{x+4}$.
83. $\frac{x-4}{x-2}$.
84. $x+3$.
85. $m=9$.
86. $p=-2$.
87. $c=20$.

EXERCISE 38. (Page 152.)

1. $(x+2)(2x+7)$.
2. $(x+5)(3x-1)$.
3. $(x+1)(5x+3)$.
4. $(2x+3)(3x-2)$.
5. $2(3x+1)(x-1)$.
6. $(2x-3)(4x-1)$.
7. $(4x+3)(2x-1)$.
8. $(4m+5)(3m-2)$.
9. $(a+1)(14a+15)$.
10. $(2p+5)(10p-3)$
11. $(p-1)(2p-3)$.
12. $(1+5m)(2-3m)$.
13. $(3a-4b)(3a-b)$.
14. $(2m-7n)(m+3n)$.
15. $(p-q)(4p+9q)$.
16. $(x-5y)(15x-2y)$.
17. $(2a+5b)(6a-b)$.
18. $(2x-7y)(5x-3y)$.
19. $\frac{x-2}{x-3}$.
20. $\frac{2x-5}{3x-5}$.
21. $(2x+2y+z)(x+y-2z)$.
22. $(a+b)^2(2a^2+ab+2b^2)$.
23. $(x-1)(x-3)(x+2)(x+4)$.
24. $(4a-5b-6)(2a+3b+8)$.
25. $7(40m-37n)(n-2m)$.
26. $(2x^2-5)(3x^2+4)$.
27. $(2x^2-3)(3x^2+1)$.
28. $(x^3-2)(5x^3+3)$.
29. $(2x^2+3y^2)(3x+y)(3x-y)$.
30. $(x-2)(2x-1)(x^2+2x+4)(4x^2+2x+1)$.
31. $(2x^2+y^2)(2x^2-y^2)(x^2+2y^2)(x^2-2y^2)$.
32. $(x-2y)(5x-3y)$.
33. $(x-3y)(5x+2y)$.
34. $(x-3)(7x-4)$.
35. $(x-4)(7x+3)$.
36. $xy(xy-5)(xy-4)$.
37. $a(2a-3b)(4a+5b)$.
38. $(a-9)(a+6)$.
39. $(x-5)(x+1)(x-2)^2$.
40. $(x^2+x-21)(x^2+x-5)$.
41. $(x+4)^2(x^2+8x+6)$.
42. $\left(\frac{x}{2}+5\right)\left(\frac{x}{2}-1\right)$.

EXERCISE 39. (Page 154.)

1. $(a-2b+3c)(a^2+4b^2+9c^2+2ab-3ac+6bc)$.
2. $(x-y-z)(x^2+y^2+z^2+xy+xz-yz)$.

3. $(a-b-1)(a^2+b^2+1+ab+a-b)$.
4. $(x+y+1)(x^2+y^2+1-xy-x-y)$.
5. $(a+b-6)(a^2+b^2+36-ab+6a+6b)$.
6. $(x-2y-4)(x^2+4y^2+16+2xy+4x-8y)$.
7. $(4a-3b+1)(16a^2+9b^2+1+12ab-4a+3b)$.
8. $(3x-5y-4)(9x^2+25y^2+16+15xy+12x-20y)$.
9. $(2a-3b-c)(4a^2+9b^2+c^2+6ab+2ac-3bc)$.
10. $\left(x-\frac{1}{x}-3\right)\left(x^2+3x+10-\frac{3}{x}+\frac{1}{x^2}\right)$.
11. $(a^2+2a-4)(a^3-2a^2+8a+16)$.
12. $2y(3x^2+y^2+3z^2+3zx+3xy+3yz)$.
13. $2(c-b)(3a^2+b^2+c^2-3ab-3ac+bc)$.
14. $9(x+2)$. **15.** $4(x+y+z)(x^2+y^2+z^2-xy-xz-yz)$.

EXERCISE 40. (Page 155.)

1. $(x^2+5x+7)(x^2+5x+3)$.
2. $(x^2+2x-7)(x^2+2x-4)$.
3. $(x^2-3x-16)(x^2-3x-12)$.
4. $(x+5)(x+6)(x^2+11x+8)$.
5. $(x^2-11x+12)(x^2-4x+12)$.
6. $(x^2-3x-16)^2$.
7. $(x-3)(2x+3)(2x^2-3x+7)$.
8. $(x+1)(3x-7)(3x^2-4x+3)$.
9. $8(x^2+x-7)(2x^2+2x-5)$.
10. $3(3x^2+x-1)(9x^2+3x+1)$.
11. $(x+2)^2(x^2+4x-6)$.
12. $(x^2+3x-13)(x^2+3x+5)$.
13. $4(2x-1)(x+4)(2x^2+7x+15)$.
14. $(9x^2-9x-17)(9x^2-9x-5)$.

EXERCISE 41. (Page 156.)

1. $(x+y+z)(x+y-z)$.
2. $(4x+2y-3z)(4x-2y+3z)$.
3. $(2x+3y-5z)(2x-3y+5z)$.
4. $(4x+6y+5z)(4x-6y+5z)$.
5. $(5x+7y-6z)(5x-7y-6z)$.
6. $(c+d+a-b)(c+d-a+b)$.
7. $(a-2b+3c-4d)(a-2b-3c+4d)$.
8. $(4a+3b-7c+5d)(4a-3b+7c+5d)$.
9. $(1-3b+2a-5c)(1-3b-2a+5c)$.
10. $(2a+3b-5c-6)(2a-3b+5c-6)$.
11. $(5c+7b-a-2)(5c-7b+a-2)$.
12. $(7a-4b-8c+1)(7a-4b+8c-1)$.
13. $(a^2+2a+3)(a^2-2a+3)$.
14. $(a^2+a+4)(a^2-a+4)$.
15. $(2a^2+2ab+3b^2)(2a^2-2ab+3b^2)$.
16. $(3a^2+5ab+4b^2)(3a^2-5ab+4b^2)$.
17. $(2a^2+5a-3)(2a^2-5a-3)$.
18. $(3a^2+3ab-4b^2)(3a^2-3ab-4b^2)$.
19. $(5a^2+7ab+3b^2)(5a^2-7ab+3b^2)$.
20. $(a^2+5ab-5b^2)(a^2-5ab-5b^2)$.

21. $(a^2+2ab-6b^2)(a^2-2ab-6b^2)$.
22. $(7a^2+2ab-4b^2)(7a^2-2ab-4b^2)$.
23. $(a^2+ab+b^2)(a^2-ab+b^2)$.
24. $(2a^2+10a+25)(2a^2-10a+25)$.
25. $(a^2+2a+2)(a^2-2a+2)$.
26. $(a^2+4a+8)(a^2-4a+8)$.
27. $9(a^2+2ab+2b^2)(a^2-2ab+2b^2)$.
28. $(8p^2+12pq+9q^2)(8p^2-12pq+9q^2)$.
29. $(a^2+a+1)(a^2-a+1)$.
30. $(a^2+a+1)(a^2-a+1)(a^4-a^2+1)$
31. $(a^2+ab+b^2)(a^2-ab+b^2)(a^4-a^2b^2+b^4)$.
32. $(5a^2+5ab+6b^2)(5a^2-5ab+6b^2)$.
33. $(3p-r)(3p+r-2q)$.
34. $(a-5c)(a+5c-6b)$.
35. $(5a-3c)(5a+3c-4b)$.
36. $(7a-3c)(7a-4b+3c)$.

EXERCISE 42. (Page 15

1. $(a+b)(a+b+1)$.
2. $(a-b)(a-b-1)$.
3. $(2a+3b)(2a+3b-4)$.
4. $(a-5b)(a-5b-3)$.
5. $(a+b-c)(a-b+c+1)$.
6. $(a-b-c)(a+b+c+1)$.
7. $(2a-b+3c)(2a+b-3c-3)$.
8. $(3a-4b-2c)(3a-4b+2c-5)$.
9. $(a-c)(a+b+c)$.
10. $(2a-3b)(2a+3b+4c)$.
11. $(3a-5b)(3a+5b-7c)$.
12. $(a+1)(a^2+1)$.
13. $(a+1)^2(a-1)$.
14. $(a-b)(a^2-ab+b^2)$.
15. $(a-2)(a^2+7a+4)$.
16. $(2a-3b)(4a+3b)(a+3b)$.
17. $(2a-3b)(4a+9b)(a+b)$.
18. $(3+4b)(9a^2-17ab+16b^2)$.
19. $(ax+by)(bx+ay)$.
20. $(a+b)(a^2+ab+b^2)$.
21. $(a-2b)(a+2b)(a^2+4b^2-5c^2)$.
22. $(2a+3b)(2a-3b)(4a^2+9b^2-7c^2)$.
23. $(a-b)^2(a^2+b^2)$.
24. $(a^2+2b^2)(a^2-5ab+2b^2)$.
25. $(a+b)(a-b)(p+3q)(p-q)$.

EXERCISE 43. (Page 160.)

1. $(a+5)(a+1)(2a-3)$.
2. $(a-3)(4a^2-3a-6)$.
3. $(a-6)(a+3)(2a-3)$.
4. $(a-1)(a-4)(2a+3)$.
5. $(a-b)(a^2+2ab+3b^2)$.
6. $(a-b)^2(a-2b)$.
7. $(a-4b)(a+2b)(a+5b)$.
8. $(2a-b)(a^2-ab+3b^2)$.
9. $(a-2b)(2a+b)(3a-4b)$.
10. $(a-b)(3a+2b)(4a+3b)$.
11. $(x+1)(2x-3)(x^2+1)$.
12. $(x-2)(2x+3)(x^2+1)$.
13. $(x-4)(x^3-2x^2-x+1)$.
14. $(x+4)(2x^3-x^2+3x-1)$.
15. $(x+2)(2x-1)(x^2+x-1)$.
16. $(x^2+x+1)(2x-1)(x+1)$.
17. $(x^2-x+1)(x^2+2x+3)$.
18. $(x-1)(x-3)(2x^2-x+1)$.
19. $(x-2)(x-3)(x^2-x+2)$.
20. $(x+3)(x-1)(x^2-2x+3)$.

EXERCISE 44. (Page 161.)

1. $\left(\frac{a}{3}+\frac{b}{2}\right)\left(\frac{a}{3}-\frac{b}{2}\right)\left(\frac{a^2}{9}+\frac{b^2}{4}\right)$.

2. $(1+a+b)(1-a-b)$.

3. $(a-b)\{a^2+b^2+ab-1\}$.

4. $(a+2b+c)(a-2b-c)$.

5. $\left(1-\frac{x}{y}\right)\left(1+\frac{x}{y}+\frac{x^2}{y^2}\right)$.

6. $(a^2+a+1)(a^2-a+1)(a^4-a^2+1)(a^8-a^4+1)$.

7. $(a-1)(a^2+a+1)(a+1)(a^2-a+1)(a^2+1)(a^6-a^2+1)$.

8. $(a+1)(a^2+6a+1)$.

9. $(x-a+b)(x-a-b)$.

10. $(a-3b-4)(a-3b+3)$.

11. $(2a+3)(8a+1)(34a^2+16a+5)$.

12. $(a^2+3a+12)(a^2+3a-7)$.

13. $4b^2$.

14. $4a^2b(a+b)$.

15. $4(6a-b)(a-6b)$.

16. $(a+b+c+d)(a+b+c-d)$.

17. $4abxy$.

18. $3a(a-7)(a+5)$.

19. $(ax+by)(x+y)$.

20. $(a-b)(a+b+c)$.

21. $(a^2+b^2)(x^2+y^2)$.

22. $(2a^2+3b^2)(3a+2b)$.

23. $(a+1)(a-1)(b+1)(b-1)$.

24. $(a^2c-1)(b^2c-1)$.

25. $3(1+2a-2b)(1-2a+2b)$.

26. $(6x-6y-1)^2$.

27. $\left(x+\frac{2}{x}\right)\left(x^2-2\right)$.

28. $(x+4)(x^2-4x+16)$.

29. $(x^2+4x+8)(x^2-4x+8)$.

30. $(x+y)(x-y)(x^2+y^2)(x^4+y^4)$.

31. $(x-y)(x^2+xy+y^2)(x^6+x^3y^3+y^6)$.

32. $(a^m-b^n)(a^{2m}+a^mb^n+b^{2n})$.

33. $3ab(3a+4b)(3a-4b)$.

34. $\left(\frac{x}{3}-\frac{y}{4}\right)\left(\frac{x^2}{9}+\frac{xy}{12}+\frac{y^2}{16}\right)$.

35. $(x-4y)(9x-5y.)$

36. $(x+5y)(3x-4y)$.

37. $(1+9a-9b)(1-10a+10b)$.

38. $(1+a)(1+b)(1+c)$.

39. $(x-\cdot1)(x^2+\cdot1x+\cdot01)$.

40. $3(2x^2+6xy+9y^2)(2x^2-6xy+9y^2)$.

41. $-(3x-7)(7x+3)$.

42. $(2x-3)(4x+7)$.

43. $2(2x-1)(x-3)$.

44. $2x(3y-x)$.

45. $(3x^2+y^2)(x^2+3y^2)$.

46. $(3p^2-2pq+3q^2)(3p^2+2pq+3q^2)$.

47. $(ax+ay-x+y)(bx+by-x+y)$.

48. $(a+b)(a-b)\left(1+\frac{1}{ab}\right)\left(1-\frac{1}{ab}\right)$.

49. $(x+a)(ax+1)$.

50. $-(3a+13b)(6a+5b)$.

51. $(a-1)(a-4)(a^2-5a-7)$.

52. $(2x^2-19xy+8y^2)(2x^2+3xy-3y^2)$.

53 $(x-3y)(x^2-3xy+9y^2)$.

54. $\left(\frac{a}{2}+b+\frac{c}{3}\right)\left(\frac{a^2}{4}+b^2+\frac{c^2}{9}-\frac{ab}{2}-\frac{ac}{6}-\frac{bc}{3}\right)$.

55. $(x+by)\left(x-\frac{1}{b}y\right)$.

56. $(2x^2+14ax+49a^2)(2x^2-14ax+49a^2)$.

57. $(x^2+4x-3)(x^2-4x-3)$.

58. $2(b+c)(a+b)$.

59. $2a(a+b+c)$.

60. $2(b-d)(a+b+c+d)$.

61. $(a+b+c+d)(2a^2+2b^2+2c^2+2d^2+ab+bc+cd+da-2ac-2bd)$.

62. $(a+b+c-d)(a+b-c+d)(c+d+a-b)(c+d-a+b)$.

63. $(3s-a-b-c)(a^2+b^2+c^2-ab-ac-bc)$.

64. $(x+8)(x-1)(x^2+7x+30)$.

65. $(a^2+6a-18)(a^2+4a-18)$.

66. $(9x^2+33x+19)^2$.

67. $(a-c)(a+b+c)(a^2+c^2-2ac+ab+bc)$.

68. $xy(xy-6)(11xy+5)$.

69. $(ax+bx+1)(ax-bx-1)$.

70. $(ac+bd+ad-bc)(ac+bd-ad+bc)$.

71. $(ac+ad+bc-bd)(ac+ad-bc+bd)$.

72. $(x-a)(x+a)(x^2+a^2)(x^4+a^4-1)$.

73. $(x^2+y^2)(x^4-x^2y^2+y^4-1)$.

74. $(y-x)(y^2+xy+x^2-x-y)$.

75. $(1+x)(1-x^2+x^3)$.

76. $(x^2-yz)(z-y)$.

77. $(a+b)(a+b-2)$.

78. $(x-y)(2x^3+2y^3+2x^2y+2xy^2+x^2+xy+y^2)$.

79. $(5p-r)(5p+r-2q)$.

80. $(x+2)(2x^2+9x+1)$.

81. $(x+1)^2(x+4)$.

82. $(x-y)(x+y)^3$.

83. $(a^2+b^2)(px-qy+rz)$.

84. $x-7$, $x+3$. 85. x^2-4x+5.

EXERCISE 45 (Page 164.)

1. $-48x(x-5)$.

2. $12x(x^2+8)$.

3. $(y-x)(x+y+z)(x^2+y^2-2xy+yz+zx)$.

4. $x(11x-10y)(x^2+22y^2)$.

5. $1+a^4+a^8$.

6. $a^4+4a^3b+6a^2b^2+4ab^3+b^4+a^2+b^2+2ab+1$.

7. $(a^2-b^2)^2$.

8. $a^8+a^4b^4+b^8$.

9. $1-a^4(1+a)^4$.

10. $2a^2b^2+2a^2c^2+2b^2c^2-a^4-b^4-c^4$.

11. $x^4+y^4+z^4-2x^2y^2-2x^2z^2-2y^2z^2$.

12. $2a^2b^2+2a^2c^2+2a^2d^2+2b^2c^2+2b^2d^2+2c^2d^2+8abcd-a^4-b^4-c^4-d^4$.

13. $a^6-14a^4b^2+49a^2b^4-36b^6$.

14. $4a^8-9a^6-4a^2+9$.

15. $a^8+16a^4b^4+256b^8$.

16. x^8-162x^4+6561.

17. x^8-32x^4+256.

18. $a+2b-c$.

19. $a^4-a^2b^2+b^4$.

20. a^4-b^4.

21. $a-1$.

22. $a-2$.

23. $a-2b$.

24. $a+b$. 25. $a^3+2a^2b+2ab^2+b^3$.

26. $a^2+b^2+c^2-2ab-ac+bc$.

27. $a^2+b^2+4c^2+2ab+2ac+2bc$. 28. $a^2+3a+13$.

29. $a^2-6ab+12b^2$. 30. a^2-2a-3.

31. $9a^2+b^2+4c^2+3ab+6ac-2bc$.

32. $25x^2+36y^2+z^2+30xy-5xz+6yz$. 33. $(a^2+ab+b^2)^2$.

34. $2x-3y+4z$. 35. $(a-b)^2+1$. 36. $6x^2-7x-3$.

37. $13x^2-23xy+13y^2$. 42. ·605. 43. ·24. 44. 266.

45. $113\frac{1}{9}$. 46. 12739. 47. 1569. 48. 1. 49. 10.

50. ·24. 51. 3·030625. 52. 132. 53. 0.

54. 330300. 55. 0. 56. 0. 57. ·000124.

58. $\frac{x(x^2+3y^2)}{4}$. 59. $4(6a^2b^2-a^4-b^4)$. 60. 288.

61. 0. 62. $a-b$. 63. $a-3b$. 64. $a+2b$.

65. $a+2$. 66. $a-3$. 67. $a-1$. 68. $a+3$.

69. $3x^2y^3$ 70. $16a^3x^3(x+a)$. 71. $2ab$. 72. ab.

73. $a+2$. 74. $a-b-c$. 75. $x+4$. 76. $a-4b$.

77. $3a-4$. 78. $x-y-z$. 79. $a^2(7a+9)$. 80. $a-2$.

81. $a-2$. 82. $a+3$. 83. a^2+1. 84. $a-2$.

85. $a-3$. 86. $2x-1$. 87. $x-y-1$.

88. $a(a-1)^2(a+1)$. 89. $(a^3+8)(a-2)$.

90. $(a-1)^3(a^2+a+1)$. 91. $(a+1)(a+2)(a+3)$.

92. $(a-3b)(a-2b)(a+b)$. 93. $(x+a)(x+4a)(x+7a)(x+13a)$.

94. $(x+1)(x+2)(x+3)$. 95. a^6-1. 96. a^8-1.

97. $a^{12}-1$. 98. $(a^2-4b^2)^2$. 99. $36a^3b^2(a-b)^5$.

100. $72(a-b)^2(a+b)^3(a^2+ab+b^2)$.

101. $60a^3b^2(a-b)^3(a+b)(a^2+ab+b^2)$.

102. $36(a^2-b^2)^3(a^2+b^2)$. 103. $a^3b(a-b)^2(a^6-b^6)$.

104. $(x-1)^3(x+1)(x^2+1)$. 105. $12a^2(a^2-9b^2)^2$.

106. $64a^6-729b^6$. 107. $(2a-1)(a+2)(3a+1)$.

108. $(a+7)(2a-3)(3a+4)(a^3-2a^2-5)$. 109. x^6-a^6.

110. $(x^2-a^2)(x^2-b^2)(x^2-c^2)$.

111. $(1-4a^2)(1+4a+4a^2-16a^4)$. 112. $\frac{x+2}{x+5}$. 113. $\frac{4(a+b)}{5(a-b)}$.

114. $\frac{x+1}{x-5}$. 115. $\frac{x-3}{x+4}$. 116. $\frac{1-4x}{1-5x}$. 117. $\frac{1-2x^2}{1+3x^2}$.

118. $\frac{3x-5}{7x-1}$. 119. $3bc$. 120. $\frac{x-a}{x^2+a}$. 121. $\frac{2x-y}{x^2-1}$.

122. $\frac{a-b-c}{a+b-c}$. 123. $\frac{(x-1)^2}{x^2+1}$. 124. $\frac{(2a+1)(a+1)}{2a^2}$.

125. $\frac{x+a-b-c}{x+b-a-c}$. **126.** $\frac{1}{x^2+x+1}$. **127.** $\frac{x^2+x+1}{(x+1)(x^2+1)}$.

128. $\frac{x^2+1}{x^4+x^2+1}$. **129.** $\frac{x^4+1}{x^8+x^4+1}$. **130.** $\frac{3x^2+1}{4x(x^2+1)}$.

131. $\frac{x+5}{x+7}$. **132.** c. **133.** $\frac{x}{y(a-b)}$. **134.** $\frac{x-1}{x-3}$.

135. $\frac{x+2}{x-5}$. **136.** $\frac{ax+by}{a^3x^3-b^3y^3}$. **137.** 1. **138.** 1.

139. 1. **140.** 1. **141.** $x+b$. **142.** 1.

SECTIONAL REVISION II. (Page 172.)

Paper 1. **1.** (i) $25x^2 \pm 40xy + 16y^2$, (ii) $10(a^2+b^2)$. **2.** 34.

4. (i) $3(a+1)(a-1)$, (ii) $(1+2a+b)(1-2a-b)$,
(iii) $4(2x-3)(3x+4)$, (iv) $(x-5)(2x-3)(x-7)$.

5. $(2a-1)(a-3)(3a-2)$. **6.** 1.

Paper 2. **1.** (i) (a) $49y^2$, (b) $132xy$; (ii) (a) 254,016, (b) 245,025.

2. 66. **3.** (i) $3n-1$, (ii) $x^4+x^3y+x^2y^2+xy^3+y^4$.

4. (i) $9(5-19x-x^2)$. **5.** (i) $x(x+7)(x-6)$, (ii) $4(2a-b)(3b-2a)$.

6. $(5a-3)(7a+2)(8a-1)$.

Paper 3. **1.** $p^2-4pq+4q^2-9r^2$ and 3255. **2.** 322.

3. $d^2=3c^2-3b^2+a^2$. **4.** $(2x-3)(3x-2)$.

5. (i) $\frac{3(a-3)}{a-5}$, (ii) $(a-8)(a-9)^3$.

6. $(a-5b)(a-3b)(a+2b)(a+7b)$.

Paper 4. **1.** (i) $(p+2)(p-3)(p+4)=p^3+(2-3+4)p^2+(-6+8-12)p-24$, (ii) $x^3+4x^2-17x-60$.

2. 52. **4.** (i) 6, (ii) 18, (iii) 23.

5. (i) $(2x^2+1)(2x^2-1)(2x^2+2x+1)(2x^2-2x+1)$,
(ii) $a^2b^2(3a-2b)(9a^2+6ab+4b^2)$,
(iii) $(7a+3b)(a+9b)$, (iv) $(x-2)(2x-1)(2x^2-5x+5)$.

6. $\frac{9(x-2y)^2}{8(x-3y)^2}$.

Paper 5. **1.** $(5m-2n)^3=(5m)^3-3(5m)^2(2n)+3(5m)(2n)^2-(2n)^3$.

2. (i) 341, (ii) 387, (iii) 4. **3.** (ii) $\frac{ku}{k-u}$.

4. (i) $2x(x-3)(x^2+3)$, (ii) $x^2(x+6)(x-5)$,
(iii) $(x-y)(x+y-z)$.

5. (i) $n=\frac{rI}{E-RI}$, (ii) -33. **6.** (i) 55, (ii) 1,

Paper 6. **1.** $27+64a^3$ and 133. **2.** p^3+3p. **3.** (i) $4(x+2y)^2$, (ii) $(x+a)(x-a)(x-b)$. **4.** (i) $16a^4$, (ii) $2m^3-9m^2+16m-14$.

5. (i) $a+2b$, (ii) $5a-3$. **6.** 1.

Paper 7. **1.** (i) $9x^2+4y^2+16z^2-12xy+24xz-16yz$, (ii) $2p^2-4q^2-4r^2-6pq+10pr-2rq$. **2.** 50. **3.** (i) 0.

4. (i) $(9a^2+12ab+8b^2)(9a^2-12ab+8b^2)$,
(ii) $(5x-5y-3)(7x-7y-4)$,
(iii) $(2x+3y)(2x-3y-3)$, (iv) $(4x^2+16x+11)^2$.

5. $(3a-1)(6a-5)(2a+3)$. **6.** 1.

Paper 8. **1.** $8m^3-n^3-27-18mn$. **2.** 246,684,600,000.

3. (i) $18x(28x^2-10x+1)$, (ii) $(2x+1)(x+3)$.
(iii) $(2a+3b)(2a+3b-4)$, (iv) $(a+1)(a+2)(2a-1)$.

5. a^2-ab+b^2. **6.** 10,000.

Paper 9. **1.** (i) ·414, (ii) $x^7-x^6a+x^5a^2-x^4a^3+x^3a^4-x^2a^5+xa^6-a^7$

3. $6m^2-7m-3$. **4.** $x-5$.

5. (i) $(a^2+2a+2)(a^2-2a+2)$, (ii) $(a+1)^2(a-1)$,
(iii) $(a^2-a+1)(a^2+a+1)(a^4-a^2+1)$. **6.** 1.

Paper 10. **1.** ·42. **2.** (i) $2a^2b(a+b)$, (ii) $(x-2)(x-3)(x-5)$.

3. $x-a$. **4.** $\frac{x^2-x+1}{x(x+9)}$. **5.** $x^{12}-y^{12}$.

EXERCISE 46. (Page 180.)

1. a^2+3a+4.
2. $a+3$.
3. $a-4$.
4. $3a-1$.
5. $2a^2+3a+7$.
6. $3a+4$.
7. $2a^2(4a^2-12a+9)$.
8. $2a^2+a-1$.
9. $a^2-2ab+4b^2$.
10. a^2-a-1.
11. a^2-2.
12. a^2+3a+5.
13. a^2-3a+1.
14. a^2+5a+1.
15. a^2-3a+5.
16. $x-2$.
17. $x-3$.
18. $(x-1)(x-2)$.
19. $(x+y)(x+4y)$.
20. $3x-2y$.
21. $2x^2-x-3$.
22. x^2-3x+4.
23. x^2-x-2.
24. x^2+2x+3.
25. x^2-2x+1.
26. x^2+2x+3.
27. $3x^2-x+1$.
28. $\frac{x-2}{x^2-x-3}$.
29. $\frac{x-1}{x+2}$.
30. $\frac{x^2+x-12}{x^2+x+2}$.
31. $\frac{2x-3}{6x+5}$.
32. $\frac{x-1}{x+1}$.
33. $\frac{x^2+x-6}{x^2-1}$.
34. $\frac{3x^2+x+2}{2x^2+x+3}$.
35. $\frac{x^2+x-12}{x^2-x-12}$.

EXERCISE 47. (Page 184.)

1. H.C.F. $= x^2-2x+3$ and L.C.M. $=(x-3)(x^3+x^2-3x+9)$.
2. H.C.F. $= x^2+3x+2$ and L.C.M. $=(x-5)(x^3-x^2-10x-8)$.
3. H.C.F. $= 2x^2-9$ and L.C.M. $=(2x-5)(6x^3+8x^2-27x-36)$.
4. H.C.F. $= 2x^2-3x+1$ and L.C.M. $=(x^2+2x-2)(2x^4-7x^3+11x^2-8x+2)$.
5. H.C.F. $= x^2+x-6$ and L.C.M. $=(x^2-3)(x^5+6x^2-49x+42)$.
6. H.C.F. $= 4x^2-3x+2$ and L.C.M. $=(2x^2-3)(16x^4-12x^3+20x^2-9x+6)$.
7. H.C.F. $= x^2-x+3$ and L.C.M. $=(x^2-x+1)(2x^4-x^3+6x^2+2x+3)$.
8. H.C.F. $= 3x^2-5ax-a^2$ and L.C.M. $=(5x-2a)(6x^4-25ax^3+26a^2x^2-a^4)$.
9. $15x^3-x^2-5x-1$.
10. $4x^4-x^3+4x-1$.
11. $1-x^4-x^6+x^7$.
12. $x^4-2x^3+10x^2-12x+24$.
13. $x^6+x^5+3x^4+11x^3+4x^2+12x+16$.
14. $x^7+x^6-x^5+2x^4-x^3-x^2+x-2$.
15. $(x^2+x-3)^2(x^2-x+3)(x^3-x+3)$.

EXERCISE 48. (Page 186.)

1. $\frac{4(x-1)}{4(x^2-1)}, \frac{3(x-1)}{4(x^2-1)}, \frac{4x}{4(x^2-1)}$.
2. $\frac{3(x-2)}{(x-3)(x-2)(x+1)}, \frac{4(x+1)}{(x-3)(x-2)(x+1)}$.
3. $\frac{a(a+b)(a^2+b^2)}{a^4-b^4}, \frac{b(a-b)(a^2+b^2)}{a^4-b^4}, \frac{ab(a^2+b^2)}{a^4-b^4}, \frac{b^2(a^2-b^2)}{a^4-b^4}$.
4. $\frac{x^2+xa+a^2}{x^4+x^2a^2+a^4}, \frac{x^2-xa+a^2}{x^4+x^2a^2+a^4}, \frac{a^2}{x^4+x^2a^2+a^4}$.
5. $\frac{a(a^2+ab+b^2)}{a^3-b^3}, \frac{a^2-b^2}{a^3-b^3}, \frac{ab}{a^3-b^3}$.
6. $\frac{x-c}{(x-a)(x-b)(x-c)}, \frac{x-b}{(x-a)(x-b)(x-c)}, \frac{x-a}{(x-a)(x-b)(x-c)}$.
7. $\frac{c-b-a}{(a+c-b)(b+c-a)(a+b-c)}, \frac{a-b-c}{(a+c-b)(b+c-a)(a+b-c)}, \frac{b-c-a}{(a+c-b)(b+c-a)(a+b-c)}$.
8. $\frac{a^2(a-b+c)}{ab(a-b-c)(a-b+c)}, \frac{b^2(a-b-c)}{ab(a-b-c)(a-b+c)}, \frac{abc}{ab(a-b-c)(a-b+c)}$.

9. $\frac{2(a+b)}{a^2-b^2}$, $\frac{-3(a+b)}{a^2-b^2}$, $\frac{4(a-b)}{a^2-b^2}$. 10. $\frac{(a+b)^2}{a^2-b^2}$, $\frac{(a-b)^2}{a^2-b^2}$, $\frac{-2ab}{a^2-b^2}$.

11. $\frac{(4a-b)(1+4ab)}{1-16a^2b^2}$, $\frac{(4a+b)(1-4ab)}{1-16a^2b^2}$, $\frac{4b(4a^2-1)}{1-16a^2b^2}$.

12. $\frac{3(x-1)}{(x-1)(x-2)(x-3)}$, $\frac{4(x-2)}{(x-1)(x-2)(x-3)}$, $\frac{-5(x-3)}{(x-1)(x-2)(x-3)}$.

EXERCISE 49. (Page 188.)

1. $\frac{5x}{(x-2)(x+3)}$. 2. $\frac{9x+1}{(x-3)(x+4)}$. 3. $\frac{4ax}{x^2-a^2}$.

4. $\left(\frac{x+y}{x-y}\right)^2$. 5. $\frac{2x}{(x+1)(x+2)}$. 6. $\frac{1}{1-x}$. 7. $\frac{1}{ab}$.

8. $\frac{1}{x-y}$. 9. $\frac{6xy}{8x^3+27y^3}$. 10. $\frac{1}{(a-b)(b-c)}$. 11. $\frac{2ab}{a^2-b^2}$.

12. $\frac{2a(a+5b)}{(a+b)^2(a-b)^2}$. 13. $\frac{2(x^2-7x+13)}{(x-2)(x-3)(x-4)(x-5)}$.

14. $\frac{1}{(x+5)(2x+3)}$. 15. $\frac{4}{(x-1)(x-5)}$. 16. 0.

17. $\frac{3}{(x+1)(x+2)(x+3)}$. 18. $\frac{1}{(x-1)(x+2)(x+3)}$.

19. $\frac{4}{(x+a)(x+13a)}$. 20. $3x$. 21. 7. 22. 1.

23. $\frac{-ab(2a+b)}{(a+b)(a-b)}$. 24. $\frac{2a^2-4a-3}{(a+1)(a-2)}$. 25. $4x^2$.

26. $\frac{x^2+2x+3}{(x^2-1)(x^2+1)}$. 27. $\frac{6(a^2-2)}{(a^2-1)(a^2-4)}$.

28. $\frac{6ab^2}{(a^2-b^2)(4a^2-b^2)}$. 29. $\frac{8a(a^2-7b^2)}{(a^2-b^2)(a^2-9b^2)}$.

30. $\frac{8(a^4-3)}{(a^4-1)(a^4-9)}$. 31. $\frac{1}{1-a^2}$. 32. $\frac{1}{1-9x^2}$.

33. $\frac{a^3+b^3}{a^3-b^3}$. 34. $\frac{8}{1-a^8}$. 35. $\frac{6}{a^{16}-1}$.

36. $\frac{-3b^2}{a(a+b)(a-b)}$. 37. $\frac{4x^3}{x^8-a^8}$. 38. $\frac{1}{a^8-1}$. 39. $\frac{-1}{x+a}$.

40. $\frac{2(a+b)}{a^3+ab+b^2}$. 41. $\frac{1}{a^2+ab+b^2}$. 42. 0. 43. $\frac{4}{(x-1)(x+3)}$.

44. $\frac{6}{(x+2)(x+3)(x+4)(x+5)}$. 45. $\frac{-48}{x(x-2)(x+2)(x+4)}$.

46. $\dfrac{16(x+4)}{(x+1)(x+3)(x+5)(x+7)}$.

47. $\dfrac{2x(a^2-b^2)}{(x^2-a^2)(x^2-b^2)}$.

48. $\dfrac{10b^2(a^4-6b^4)}{(a^4-4b^4)(a^4-9b^4)}$.

EXERCISE 50. (Page 194.)

1. x. 2. 1. 3. 1. 4. $\dfrac{x+1}{x-1}$. 5. $\dfrac{2xy}{x^2+y^2}$. 6. 1.

7. $\dfrac{x^2-y^2}{x^4+x^2y^2+y^4}$. 8. $\dfrac{x+1}{x+3}$. 9. $\dfrac{x+a+x^2-ax}{x-a}$.

10. $\dfrac{x^4-y^4}{2xy}$. 11. 1. 12. $\dfrac{x-z}{1+xz}$. 13. $\dfrac{y-z}{1+yz}$. 14 1.

15. a^2-2a+1. 16. $-\dfrac{1}{a}$. 17. $\dfrac{b(2a+b)}{a^2(a^2+b^2)}$. 18. $\dfrac{xy}{x+y}$.

19. $\dfrac{x^2}{x^4-x^2+1}$. 20. x^2 21. $\dfrac{6(1-x)}{4-x}$. 22. $\dfrac{a}{a-b}$. 23. a.

24. $\dfrac{ab^3+a^2-ab-b^2}{ab^3+a^2-ab-b^2-b^3-a+b}$. 25. $1+x+x^2$. 26. $\dfrac{x(1+x+x^3)}{1+x^2}$.

27. $\dfrac{x^3}{y^2}$. 28. $\dfrac{x(2x^2-3x+6)}{6}$. 29 $-x$. 30. $\dfrac{2x^4y^5}{x^8+x^4y^4+y^8}$.

31. $1+\dfrac{4}{x+3}$. 32. $1+\dfrac{10}{x-3}$. 33. $2+\dfrac{7}{x-1}$.

34. $3-\dfrac{10}{x-2}$. 35. $3+\dfrac{4}{x-3}$. 36. $2-\dfrac{9}{2x-1}$.

37. $2-\dfrac{11}{2x+1}$. 38. $2-\dfrac{5}{3x+2}$. 39. $2-\dfrac{3}{3x-2}$.

40. $x^3-x^2a+xa^2-a^3+\dfrac{a^4-1}{x+a}$. 41. $2x+\dfrac{2x+1}{2x^2+2x+3}$.

51. 1·111.

EXERCISE 51. (Page 201.)

1. $x=\frac{6}{5}$, $y=-\frac{2}{5}$. 2. $x=1$, $y=-2$. 3. $x=3$, $y=2$.

4. $x=4$, $y=3$ 5. $x=y=1$. 6. $x=5$, $y=1$.

7. $x=7$, $y=4$. 8. $x=4$, $y=1$. 9. $x=\dfrac{b^2+ac}{a^2+b}$,

$y=\dfrac{ab-c}{a^2+b}$. 10. $x=8$, $y=5$. 11. $x=1$, $y=3$.

12. $x=1$, $y=2$. 13. $x=5$, $y=6$. 14. $x=y=-1$.

15. $x=1$, $y=-\frac{1}{2}$. 16. $x=6$, $y=-\frac{2}{3}\frac{8}{}$. 17. $x=5$, $y=3$.

18. $x=3$, $y=2$. 19. $x=6$, $y=4$.

20. $x=\dfrac{c(c-b)}{a(a-b)}$, $y=\dfrac{c(a-c)}{b(a-b)}$.

21. $x=10,\ y=4.$ 22. $x=19,\ y=7.$ 23. $x=5,\ y=6.$
24. $x=4,\ y=2.$ 25. $x=3,\ y=2.$ 26. $x=14,\ y=15.$
27. $x=1,\ y=2.$ 28. $x=5,\ y=3.$ 29. $x=\cdot02,\ y=2\cdot9.$
30. $x=18,\ y=48.$ 31. $x=4,\ y=2.$ 32. $x=3,\ y=2.$
33. $x=3,\ y=2.$ 34. $x=8,\ y=5.$ 35. $x=3,\ y=2.$
36. $x=40,\ y=60.$ 37. $x=\frac{4}{3},\ y=\frac{5}{2}.$ 38. $x=5,\ y=2.$

39. $x=2,\ y=5.$ 40. $x=\dfrac{ac-bd-c^2}{a^2-ac-ab+b^2},\ y=\dfrac{-ac+bc+ad}{a^2-ac-ab+b^2}.$

41. $x=1,\ y=1.$ 42. $x=3,\ y=4.$ 43. $x=4,\ y=10.$
44. $x=3,\ y=4.$ 45. $x=\frac{1}{5},\ y=\frac{1}{7}.$ 46. $x=\frac{1}{2},\ y=1.$

47. $x=\dfrac{mp-nq}{ap},\ y=\dfrac{mp-nq}{aq}.$ 48. $x=2,\ y=0.$

49. $x=5,\ y=3.$ 50. $x=-3,\ y=4.$ 51. $x=-1,\ y=2.$
52. $x=10,\ y=15.$ 53. $x=4,\ y=6.$

54. $x=\dfrac{am-bn}{a^2-b^2},\ y=\dfrac{an-bm}{a^2-b^2}.$ 55. $x=7,\ y=3.$

EXERCISE 52. (Page 208.)

1. $x=6,\ y=11,\ z=6.$ 2. $x=1,\ y=2,\ z=3.$
3. $x=-11,\ y=-8,\ z=-6.$ 4. $x=2,\ y=3,\ z=1.$
5. $x=3,\ y=2,\ z=5.$ 6. $x=2,\ y=3,\ z=4.$
7. $x=1,\ y=\frac{1}{2},\ z=\frac{1}{3}.$ 8. $x=\frac{1}{3},\ y=\frac{1}{4},\ z=\frac{1}{5}.$
9. $x=11,\ y=9,\ z=7.$ 10. $x=a+c-b,\ y=a+b-c,\ z=b+c-a.$

11. $x=\frac{2}{3},\ y=-2,\ z=\frac{2}{5}.$

12. $x=\dfrac{2}{a+c-b},\ y=\dfrac{2}{a+b-c},\ z=\dfrac{2}{b+c-a}.$

13. $x=14,\ y=12,\ z=10.$ 14. $x=4,\ y=3,\ z=2.$

15. $x=\dfrac{b^2+c^2-a^2}{2bc},\ y=\dfrac{c^2+a^2-b^2}{2ca},\ z=\dfrac{a^2+b^2-c^2}{2ab}.$

16. $x=y=z=5.$ 17. $x=y=z=1.$ 18. $x=2,\ y=4,\ z=6.$

19. $x=\dfrac{2}{a+c-b},\ y=\dfrac{2}{a+b-c},\ z=\dfrac{2}{b+c-a}.$

20. $x=abc,\ y=ab+bc+ca,\ z=a+b+c.$
21. $x=-(ab+bc+ca),\ y=a+b+c,\ z=1.$
22. $x=5,\ y=4,\ z=6.$ 23. $x=4,\ y=5,\ z=6.$
24. $x=7,\ y=5,\ z=3.$ 25. $x=7,\ y=4.$
26. $x=12,\ y=5.$ 27. $x=15,\ y=8.$
28. $x=12,\ y=7.$ 29. $x=8,\ y=7.$

30. $x=11,\ y=7$. 31. $x=\dfrac{a+\sqrt{a^2-4b^2}}{2}$, $y=\dfrac{a-\sqrt{a^2-4b^2}}{2}$.

32. $x=\dfrac{\sqrt{m^2+4n^2}+m}{2}$, $y=\dfrac{\sqrt{m^2+4n^2}-m}{2}$.

33. $x=3,\ y=4$, or $x=4,\ y=3$. 34. $x=2,\ y=5$.

EXERCISE 53. (Page 213.)

1. 23, 16. 2. 15, 13. 3. 21, 12. 4 18, 15.
5. ·32, ·24. 6. 12″, 15″. 7. 15, 22, 40. 8. 18, 12.
9. 20, 12, 8. 10. Rs. 140, Rs. 180.
11. $x=160,\ y=180,\ z=210$. 12. Rs. 70, Rs. 50.
13. Rs. 3 as. 8, Rs. 5 as. 8. 14. Rs. 7, Rs. 4.
15. $\frac{7}{9}$. 16. $\frac{2}{3}$. 17. $\frac{8}{15}$. 18. $\frac{5}{11}$. 19. $\frac{8}{12}$.
20. Rs. 350, Rs. 460. 21. Rs. 22, Rs. 24.
22. Rs. 535, Rs. 315. 23. 79. 24. 15 shillings; 18 pence.
25. Tea 3*s*. 6*d*. per lb.; butter 2*s*. 6*d*. per lb. 26. 63.
27. 54. 28. 426. 29. 648. 30. 253.
31. 40 miles. 32. $8\frac{1}{2}$ hours. 33. 4 miles, $3\frac{1}{2}$ miles per hour.
34. $3\frac{1}{2}$ miles, $1\frac{1}{2}$ miles. 35. 22 miles, 132 miles.
36. 3 miles, 1 mile an hour 37. 8 miles and 3 miles an hour.
38. 288 sq. ft. 39. $x=40°,\ y=35°$. 40. 90 lbs., 30 lbs.
41. $4\frac{1}{2}s.$, $7\frac{1}{2}s.$, 10*s*. 42. $2\frac{1}{2}$ mds., $3\frac{1}{2}$ mds.; limit of concession 20 seers. 43. Rs. 55. 44. 48 ft., 36 ft.
45. 20 ft., 8 ft. 46. 80 ft., 48 ft. 47. 4″.
48. 36 ft., 25 ft., 11 ft.

SECTIONAL REVISION III. (Page 221.)

Paper 1. 2. (i) $\dfrac{1}{a-b}$. (ii) $2(a+b)$. 3. $x=4,\ y=6$.
4. 6, 7. 5. 52.

Paper 2. 1. $3x-2$. 2. (i) $x^2-xa+a^2-\dfrac{a^3+1}{x+a}$.

(ii) $\dfrac{2}{x^2(x+2)(x-2)}$. 3. $x=3,\ y=5$. 4. *A* 32 years; *B* 20 years. 5. Rs. $6n+44$.

Paper 3. 1. x^2-2x+1. 2. $\dfrac{17x}{(x-3)(x-7)(x+4)}$. 3. $x=3,\ y=2$.
4. 87. 5. $(2x+3)(3x-8)$.

Paper 4. 1. $\dfrac{2x^2+7x+3}{x^2+3x+2}$. 2. $\dfrac{8x^3}{1-x^6}$. 3. $x=4,\ y=5,\ z=3$.

4. 15 miles per hour, 90 miles. 5. 27.

Paper 5. 2. $\frac{34xy}{49x^2-y^2}$. 3. $x=4, y=3, z=2$. 4. 25 miles.

5. (i) $(x^2-3x-6)(x^2-3x+4)$, (ii) $(9x^2+12x+8)(9x^2-12x+8)$.

Paper 6. 2. $\frac{1}{x-1}$. 3. $x=3, y=4$. 4. Rs. 480, Rs. 960.

5. 143.

Paper 7. 1. $3x^2+8xy+4y^2$. 2. $\frac{2}{x}$. 4. Tea 2*s*. 6*d*. per lb., coffee 1*s*. 4*d*. per lb. 5. -5.

Paper 8. 1. H.C.F. $=x^2+1$, L.C.M. $=x^3(x^2+1)(x^2-1)(x-2)$.

2. $\frac{(x+1)^2}{x+2}$. 3. $x=4, y=5$. 4. 2 yards. 5. $(a-3)(2a^2-a+1)$.

Paper 9. 1. $(x^2+8ax-2a^2)(3x-7a)(7x-4a)$. 2. 2.

3. $x=2, y=3, z=4$. 4. 48 persons; Rs. 2 8 as. each.

5. +11.

Paper 10. 1. a^3+4a^2-3. 2. $\frac{b(2a+b)}{a^2(a^2+b^2)}$. 3. $x=5, y=4$; or $x=4, y=5$.

4. $x=47, y=27$. 5. $(a+b)(a^2-ab+b^2)(p-q)(p-2q)$.

EXERCISE 54. (Page 227.)

1. 10, 26, 17, 25.

2. (i) 5, (ii) 10, (iii) 13, (iv) 17.

5. 5.

6. 42.

EXERCISE 56. (Page 239.)

5. (i) $y=\frac{x}{2}-\frac{5}{2}$. (ii) $\frac{x}{5}+\frac{y}{-\frac{5}{2}}=1$. (iii) $x-2y-5=0$.

6. (i) $y=-2x+8$. (ii) $\frac{x}{4}+\frac{y}{8}=1$. (iii) $2x+y-8=0$.

7. (i) $y=\frac{3}{4}x-\frac{5}{2}$. (ii) $\frac{x}{\frac{10}{3}}+\frac{y}{-\frac{5}{2}}=1$. (iii) $3x-4y-10=0$.

8. (i) $y=\frac{1}{3}x-\frac{3}{8}$. (ii) $\frac{x}{3}+\frac{y}{-\frac{3}{8}}=1$. (iii) $x-8y-3=0$.

9. (i) $y=\frac{4}{3}x+4$. (ii) $\frac{x}{-3}+\frac{y}{4}=1$. (iii) $4x-3y+12=0$.

10. (i) $y=-\frac{16}{25}x+\frac{4}{5}$. (ii) $\frac{x}{\frac{5}{4}}+\frac{y}{\frac{4}{5}}=1$. (iii) $16x+25y-20=0$.

11. $y=0x+3$.

12. $0y=-x+5$. **13.** $y=0x+0$. **14.** $0y=x+0$.
34. 34·225. **35.** $x=16$. **36.** 11, −4·3, 6·5.
43. $3x-2y-2=0$. **44.** $x-y+8=0$. **45.** $4x=5y+20$.
46. $4x+3y=0$. **47.** $m=-\frac{1}{2}$, $c=\frac{13}{2}$. **48.** $y+x=0$.
49. No. **50.** $3x-2y=1$, $2x-y=2$, (3, 4).

EXERCISE 57. (Page 242.)

1. $x=3$, $y=4$. **2.** $x=5$, $y=4$. **3.** $x=4$, $y=5$.
4. $x=4$, $y=5$. **5.** $x=2$, $y=3$. **6.** $x=3$, $y=2$.
7. $x=-5$, $y=-2$. **8.** $x=2$, $y=-5$. **9.** $x=4$, $y=3$.
10. $x=4$, $y=-3$. **11.** $x=-3$, $y=4$. **12.** $x=3$, $y=4$.
13. $x=5$, $y=6$. **14.** $x=4$, $y=5$. **15.** $x=2$, $y=3$.
16. (−5, −5), (−5, 9), (2, 2), length of sides $7\sqrt{2}$, $7\sqrt{2}$, 14.

EXERCISE 58. (Page 248.)

1. $y=\frac{3}{2}x$. **4.** 72·6. **5.** (i) 53·13 gallons. (iii) 10·6 cu. ft.
7. 11·7 ft. per second, 36·7 ft. per second, 16·4 miles per hour. 23·9 miles. **8.** 102·7, 10-20 A.M. and 7-20 P.M.
9. 22nd June to 6th July. **10.** 13, 14·4, 20·6. **11.** 27″, 22″ (Approx.)
12. Rs. 26 as. 12, Rs. 33. **13.** 38·6°. **14.** 324 ft., 81 ft.

EXERCISE 59. (Page 256.)

6. (i) The latter is greater; (ii) The latter is greater.
7. −57. **11.** 23. **12.** $\frac{-ab}{a+b}$. **13.** $\frac{-ab(a+b)}{(a^2+ab+b^2)}$.
14. 28, 32. **15.** 15, 20. **16.** $49\frac{1}{2}$, $38\frac{1}{2}$. **17.** 21, 35.
18. $\frac{bx}{a}$. **19.** $\frac{7}{48}$. **20.** $1\frac{9}{29}$. **21.** $\frac{3}{4}$. **22.** $\pm\frac{5}{4}$.
23. $\frac{6}{5}, -\frac{2}{11}$. **24.** The former is greater. **25.** The former is greater.
26. 1. **27.** 3, 12. **28.** 45, 36. **29.** 9 : 10. **30.** $\frac{pq}{p+q}$.
31. $\sqrt{2}:\sqrt{5}$. **32.** 28000, 8400. **33.** $-4\frac{1}{2}$. **34.** $1\frac{1}{12}$.
35. $-1\frac{1}{2}$. **36.** $-\frac{2}{7a}$. **37.** $\frac{1}{10}$. **38.** 2.

EXERCISE 60. (Page 263.)

1. 21. **2.** 20. **3.** 40. **4.** $\frac{1}{6}$. **5.** $4a^2b^2$. **6.** 12.
7. 35. **8.** 24. **9.** 132. **10.** 3. **11.** xy.

12. $(a+b)^2(a-b)^2$. **13.** 8. **14.** $12\frac{4}{5}$. **15.** $2\frac{2}{5}$.

16. $\frac{(x+y)^2}{x-y}$. **17.** 3. **18.** 5. **19.** 6. **21.** 4, 36

22. 22. **24.** 12, 21, 24, 42, 36, 63, 48, 84.

25. Copper 60 lbs., zinc 120 lbs., lead 160 lbs., tin 200 lbs.

26. 13 : 35. **27.** 3 : 4.

EXERCISE 61. (Page 268.)

25. 2. **26.** 2.

EXERCISE 62. (Page 273.)

11. $\frac{x+y}{11}=\frac{x-y}{5}=\frac{15x-12y}{84}$. **12.** $\frac{x+y}{10}=\frac{x-y}{4}=\frac{3x+5y}{36}$.

EXERCISE 64. (Page 278.)

1. $\frac{1}{2}$. **2.** $\frac{2}{7}\frac{5}{2}$. **3.** $4\frac{1}{5}$. **4.** $\frac{1}{30}$. **5.** $\frac{F}{ab}$.

6. 3. **7.** 7. **8.** 6. **9.** 12. **10.** $\frac{5}{13}$.

11. $-7\frac{6}{11}$. **12.** $\frac{2ab}{a+b}$. **13.** $\frac{-2ab}{a^2+b^2}$. **14.** 14. **15.** $\frac{-1}{a+b}$.

16. 1. **17.** $-2\frac{1}{2}$. **18.** -6. **19.** $-1\frac{1}{2}$. **20.** 4.

21. 6. **22.** $-5\frac{1}{3}$. **23.** $\frac{a^2+b^2}{a+b}$. **24.** $\frac{1}{12}$. **25.** 7.

26. 7. **27.** 13. **28.** 0. **29.** $-1\frac{1}{2}$. **30.** $-1\frac{3}{5}$.

31. $2\frac{2}{2}\frac{1}{9}$. **32.** $-3\frac{1}{3}\frac{1}{8}$. **33.** $\frac{-2ab}{a+b}$. **34.** $\frac{2(a+b)}{a+2b}$.

35. $\frac{ab(a+b-2c)}{a^2-ac-bc+b^2}$. **36.** $\frac{ab}{c-a-b}$. **37.** $2\frac{3}{4}$. **38.** $3\frac{4}{7}$.

39. $-\frac{2}{13}$. **40.** $-\frac{4}{11}\frac{9}{4}$. **41.** $\frac{3}{4}$. **42.** -1. **43.** $-\frac{5}{2}\frac{0}{6}\frac{0}{3}$.

44. $\frac{a^2-bc}{b+c-2a}$. **45.** $-2\frac{1}{2}$. **46.** $3a$. **47.** $\frac{-7}{26}$.

EXERCISE 65. (Page 283.)

1. 3. **2.** 2. **3.** 3. **4.** $\frac{1}{2}$. **5.** 1. **6.** $\frac{ac-bd}{ad-bc}$.

7. $-\frac{7}{12}$. **8.** $-\frac{5}{16}$. **9.** $-\frac{1}{20}$. **10.** $-1\frac{1}{3}$. **11.** 0.

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12. $\frac{ac^2-b^3}{b^2+ab-c^2-a^2}$. 13. 3. 14. 5. 15. 6. 16. 7.
17. 8. 18. 7. 19. $-\frac{13}{18}$. 20. $-\frac{44}{105}$. 21. $\frac{2}{19}$.
22. $\frac{-b^3}{(a-b)^2}$. 23. 4. 24. 4. 25. $-\frac{3}{5}$. 26. $1\frac{4}{23}$.
27. $2\frac{1}{5}$. 28. $1\frac{1}{3}$. 29. $-1\frac{2}{5}$. 30. -4. 31. $\frac{7}{9}$.
32. $\frac{2}{5}$. 33. $-1\frac{2}{3}$. 34. $-\frac{10}{11}$. 35. $\frac{8}{17}$. 36. $-\frac{8}{7}$.
37. $-2\frac{3}{5}$. 38. $-1\frac{4}{7}$. 39. $1\frac{5}{7}$. 40. $\frac{1}{5}$. 41. $\frac{ab}{a+b}$.
42. $-\frac{3}{4}$. 43. -4. 44. $\frac{1}{2}$. 45. $8\frac{1}{2}$. 46. 8.
47. $2\frac{1}{2}$. 48. $-\frac{5}{4}$. 49. 4. 50. $4\frac{1}{2}$. 51. $-\frac{5}{6}$.
52. $-3\frac{1}{2}$. 53. 1.

EXERCISE 66. (Page 289.)

1. $\frac{8}{29}$. 2. $\frac{13}{22}$. 3. $\frac{13}{37}$. 4. $\frac{1}{5}$. 5. $-\frac{5}{74}$.
6. $\frac{9}{38}$. 7. $\frac{q^3-pr^2}{p^2-q^2-pq+r^2}$. 8. $\frac{1}{ab+3a-b-2}$.
9. $-\frac{23}{7}$. 10. $-4\frac{1}{3}$. 11. $-3\frac{5}{7}$. 12. $-4\frac{6}{7}$.
13. $-3\frac{10}{11}$. 14. -1. 15. $-2\frac{2}{5}$. 16. $-3\frac{3}{4}$.
17. $4\frac{4}{9}$. 18. $1\frac{7}{8}$. 19. $\frac{20}{21}$. 20. -4.
21. $\frac{2bc}{a(b+c)}$. 22. $\frac{b-c}{2a}$. 23. $\frac{3}{7}$. 24. -1.
25. $\frac{8}{3}$. 26. $-\frac{1}{4}$. 27. $2\frac{1}{2}$. 28. $1\frac{3}{5}$.
29. 1. 30. 4.

EXERCISE 67. (Page 295.)

1. $a+b$. 2. $a+b$. 3. $\frac{2ab}{a+b}$. 4. $\frac{a+b}{2}$.
5. $\frac{a^2+4ab+b^2}{2(a+b)}$. 6. $\frac{6a}{7}$. 7. $\frac{1}{ab}$. 8. 1. 9. ab.
10. $\frac{a+b+c}{3}$. 11. a or b. 12. $a+b+1$. 13. $ab+bc+ca$.
14. $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$. 15. $a(a+b+c)$. 16. $a^3+b^3+c^3$. 17. $\frac{a+b}{2}$.
18. $-\frac{a+b}{2}$. 19. $\frac{2}{a+b}$. 20. $\frac{b(a+c-b)}{a}$. 21. $\frac{2c-b-a}{2ab-ac-bc}$.
22. 1. 23. $\frac{a+b}{2}$. 24. $\frac{2ab}{b-a}$. 25. $\frac{br-cq}{cp-ar}$.
26. $-b$. 27. 25. 28. $-\frac{a+b}{2}$. 29. $\frac{a}{3}$.

EXERCISE 68. (Page 300.)

1. 1 : 3 : 5. 2. 3 : 4 : 2. 3. 3 : 4 : 5. 4. 2 : 8 : 5.

5. $(ab-c^2) : (cb-a^2) : (ac-b^2)$. 6. $x=7, y=10, z=8$.

7. $x=2, y=-3, z=1$. 8. $x=2, y=-4, z=6$.

9. $x=3, y=6, z=9$. 10. $x=8, y=12, z=20$.

11. $x=3, y=4, z=5$. 12. $x=3, y=2, z=1$.

13. $x=1, y=2, z=3$. 14. $x=b-c, y=c-a, z=a-b$.

15. $x=8, y=6, z=4$. 16. $x=3, y=9, z=15$.

17. $x=3, y=1\frac{1}{2}$. 18. $x=7, y=9$.

19. $x=1, y=-\frac{1}{2}$. 20. $x=3, y=2$. 21. $x=7, y=9$.

22. $x=\dfrac{2p^2-3}{3p+5}, \ y=\dfrac{10p+9}{3p+5}$. 23. $x=\dfrac{lp}{l^2+m^2}, \ y=\dfrac{mp}{l^2+m^2}$.

24. $x=\dfrac{1-b}{1+a}, \ y=\dfrac{b-1}{a+1}$. 25. $x=\dfrac{ab(ad-bc)}{a^2-b^2}, \ y=\dfrac{ab(ac-bd)}{a^2-b^2}$.

26. 5. 27. $x=-2, y=1$. 28. $x=\frac{1}{2}, y=\frac{1}{3}$.

29. $x=\frac{1}{2}, y=\frac{1}{5}$. 30. $x=-1, y=-\frac{1}{2}$. 31. $x=3, y=\frac{1}{2}$.

32. $x=\frac{1}{3}, y=\frac{1}{5}$. 33. $x=y=1$. 34. $x=\frac{1}{2}, y=1$.

35. $x=\frac{1}{15}, y=18$. 36. $x=\dfrac{a^2-b^2}{am-bn}, \ y=\dfrac{a^2-b^2}{an-bm}$.

37. $x=\dfrac{1}{a}, \ y=\dfrac{1}{b}$. 38. $x=\dfrac{62}{45}, \ y=-\dfrac{46}{45}$. 39. $x=-2, y=1$.

40. $x=\frac{1}{5}, y=\frac{1}{6}$. 41. $x=\frac{2}{7}, y=4$.

EXERCISE 69. (Page 304.)

1. $x=-3, y=3, z=1$. 2. $x=3, y=2, z=1$.

3. $x=-7, y=3, z=8$. 4. $x=4, y=5, z=6$.

5. $x=y=z=1$. 6. $x=1, y=\frac{1}{2}, z=\frac{1}{3}$.

7. $x=6\frac{1}{3}, y=3, z=-\frac{3}{20}$. 8. $x=10, y=1, z=9$.

SECTIONAL REVISION IV. (Page 305.)

Paper 1. 5. (i) $\dfrac{2c-b-a}{2ab-bc-ac}$. (ii) $2a, \ \dfrac{a}{2}$. 6. 17·49.

Paper 2. 1. $c : 8a$. 2. -2. 5. (i) $\frac{3}{4}$; (ii) 1. 6. $3y+2x=6$.

Paper 3. 1. 22. 2. $\frac{39}{38}$. 3. 2 : 1. 4. $-\dfrac{2}{ab}(a^2+ab+b^2)$.

5. (i) $\frac{1}{3}$; (ii) $-\frac{23}{4}$. 6. $5x+4y=14$. 7. $x=3, y=2$.

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Paper 4. 1. 4 : 1. 2. 20 : 27. 5. (i) $\frac{15}{8}$, (ii) $-\frac{47}{11}$.
7. $x=4, y=3, z=2$.

Paper 5. 2. 4. 5. (i) $\frac{4a}{5}$; (ii) 1. 6. $6x+11y=14$,
$4x+3y=18$, $x+4y=11$.

Paper 6. 2. 72 and 27. 5. (i) 7; (ii) 7.

Paper 7. 1. $\frac{5}{8}$. 2. 0. 5. (i) -4; (ii) $1\frac{3}{4}$.

Paper 8. 1. 30,000, 18,600. 2. (i) 28; (ii) $22\frac{1}{2}$; (iii) $37\frac{1}{2}$.
5. (i) $-2\frac{2}{3}$, (ii) 8. 6. 20 miles. 7. (3, 2) $(-4, 5)$, $(1, -3)$.

EXERCISE 70. (Page 313.)

1. $\sqrt[5]{a^3}$. 2. $\frac{1}{\sqrt[5]{x^3}}$. 3. $\frac{6}{\sqrt[3]{x^2}\sqrt{a}}$. 4. $\frac{9}{\sqrt[4]{p^{11}}}$.

5. $\frac{\sqrt{a^3}}{\sqrt[3]{m^2}}$. 6. $4\sqrt[3]{x^2}$. 7. $\sqrt[5]{m^3x^2}$. 8. $\frac{\sqrt{a^5}}{\sqrt[2m]{x^7}}$.

9. $\sqrt[2m]{x^{11}}$. 10. $x^{\frac{3}{5}}$. 11. $\frac{1}{m^{\frac{3}{2}}}$. 12. $m^{\frac{2}{3}}$. 13. $a^{\frac{4}{3}}$.

14. $x^{\frac{5}{3}}$. 15. $\frac{1}{m^{\frac{1}{2}}}$. 16. $\frac{1}{2}$. 17. 4. 18. 8. 19. 16. 20. $\frac{1}{8}$.

21. $\frac{1}{9}$. 22. 25. 23. $\frac{1}{27}$. 24. $\frac{1}{8}$. 25. $\frac{1}{49}$. 26. 9. 27. 81.

28. 36. 29. 8. 30. $\frac{1}{50}$. 31. 11. 32. 5. 33. $(25)^3$.

34. $\frac{x^2+1}{x(x-1)}$. 35. a^{m-2n}.

EXERCISE 71. (Page 316.)

1. $\frac{1}{x^2}$. 2. $\frac{y^{\frac{15}{2}}}{x^{\frac{9}{16}}}$. 3. xb^6. 4. $\frac{1}{x^8y^{\frac{5}{3}}}$. 5. m^6n^8.

6. $\frac{b^3}{a^6}$. 7. $m^{\frac{21}{16}}$. 8. $\frac{y}{x^2}$. 9. $\frac{1}{a^{\frac{1}{4}}}$. 10. $\frac{b}{a^{\frac{1}{2}}}$.

11. $\frac{9a^2b^2}{16}$. 12. $\frac{9}{25a^2b^2}$. 13. 1. 14. $\frac{x^{\frac{3}{2}}z^{\frac{1}{2}}}{y^{\frac{1}{3}}}$. 15. x^{2abc}.

16. 1. 17. The latter is greater by 448.

EXERCISE 72. (Page 318.)

1. (i) $5x^{-2}+3x^{-1}+1-7x+4x^2$. (ii) $6x^{-3}+4x^{-2}+7x^{-1}+3+5x^3$.
(iii) $-x^{-\frac{1}{n}}+6x^{-\frac{1}{2n}}+4+3x^{\frac{1}{2n}}+5x^{\frac{1}{n}}+x$.
(iv) $6x^{-\frac{1}{3n}}-2+5x^{\frac{1}{2n}}+7x^{\frac{1}{n}}+x$.

2. (i) $x^{-2}y^2+2+x^2y^{-2}$; (ii) $x^{-\frac{2}{3}}y^{\frac{2}{3}}-1+x^{\frac{2}{3}}y^{-\frac{2}{3}}$;
(iii) $(7x^3+1)-5x^2y^{-1}-xy^{-2}+3y^{-3}$. 3. $a+a^{-1}$. 4. $a^{\frac{3}{2}}+b^{\frac{3}{2}}$.

5. $a^2+8ac^{\frac{1}{3}}+16c^{\frac{2}{3}}-9b$. 6. $a-2a^{\frac{1}{2}}+1$.

7. $x^{-2}+x^{-1}y^{-1}+y^{-2}$. 8. $a+b+c-3(abc)^{\frac{1}{3}}$.

9. $a^{\frac{5}{6}}-5a^{\frac{2}{3}}b^{\frac{1}{6}}+10a^{\frac{1}{2}}b^{\frac{1}{3}}-10a^{\frac{1}{3}}b^{\frac{1}{2}}+5a^{\frac{1}{6}}b^{\frac{2}{3}}-b^{\frac{5}{6}}$.

10. $x+x^{\frac{3}{4}}y^{-\frac{1}{4}}-x^{\frac{1}{4}}y^{-\frac{3}{4}}-y^{-1}$. 11. $a^{\frac{1}{3}}b^{-\frac{1}{3}}-a^{\frac{1}{3}}-b^{-\frac{1}{3}}+1$.

12. $x^{\frac{2}{3}}-x^{\frac{1}{3}}a^{\frac{1}{3}}+a^{\frac{2}{3}}$. 13. $x^{\frac{4}{3}}y^{-\frac{4}{3}}+x^{\frac{2}{3}}y^{-\frac{2}{3}}+x^{-\frac{2}{3}}y^{\frac{2}{3}}+x^{-\frac{4}{3}}y^{\frac{4}{3}}$.

14. $1-x^{-\frac{1}{4}}+x^{\frac{3}{4}}$. 15. $x^{\frac{4}{5}}-x^{\frac{3}{5}}a^{\frac{1}{5}}+x^{\frac{2}{5}}a^{\frac{2}{5}}-x^{\frac{1}{5}}a^{\frac{3}{5}}+a^{\frac{4}{5}}$.

16. $x^{-\frac{2}{3}}-x^{-\frac{1}{3}}y^{-\frac{1}{3}}+y^{-\frac{2}{3}}$. 17. $4a+b^{-1}+c^{\frac{2}{3}}-2a^{\frac{1}{2}}b^{-\frac{1}{2}}+2a^{\frac{1}{2}}c^{\frac{1}{3}}+b^{-\frac{1}{2}}c^{\frac{1}{3}}$.

18. $a^{\frac{2}{3}}+2a^{\frac{1}{3}}b^{\frac{1}{3}}+b^{\frac{2}{3}}$. 19. a^2+2+a^{-2}.

20. $a^2+2a^{\frac{3}{2}}-2a^{\frac{4}{3}}+a-2a^{\frac{5}{6}}+a^{\frac{2}{3}}$.

21. $a^{\frac{2}{3}}-4a^{\frac{5}{6}}+4a+2a^{\frac{7}{6}}-4a^{\frac{4}{3}}+a^{\frac{5}{3}}$. 22. $a^{\frac{2}{3}}+2a^{\frac{1}{3}}+3+2a^{-\frac{1}{3}}+a^{-\frac{2}{3}}$.

23. $a^{\frac{3}{2}}-2a^{\frac{5}{4}}b^{-\frac{1}{4}}+ab^{-\frac{1}{2}}+2a^{\frac{3}{4}}b^{-\frac{1}{2}}-2a^{\frac{1}{2}}b^{-\frac{3}{4}}+b^{-1}$.

25. $a^{3n}-b^{-3n}$. 26. $a^n+a^{\frac{n}{2}}b^{-\frac{n}{2}}+b^{-n}$.

27. a^3-b^3. 28. $1-x^2$. 29. x^{2^n+3}. 30. x^{3^n+3}.

31. $\frac{1}{3}$. 32. $am-1$. 33. $x^{2^n}-a^{2^n}$.

34. $a^{2^{n-1}}-b^{2^{n-1}}$. 35. The latter is greater by 65,280. 36. 3^{2^4}.

37. $\frac{2}{3}$. 38. 7. 39. $\frac{1}{2^{n-1}}$. 40. 1·6. 41. $\frac{1}{5}$. 42. $\frac{1}{9^{n+1}}$.

45. $\left(\frac{p}{q}\right)^{m+n}$ 46. $1+x^{-1}$. 47. $x^2+xy^{-1}+y^{-2}$.

48. $\frac{1}{(1-x)^n}$. 49. 1. 50. 8. 51. 1.

52. $-(b^{-1}-a^{-1})(a^{-1}-c^{-1})(c^{-1}-b^{-1})(a+b+c)$. 55. 1.

EXERCISE 73. (Page 325.)

1. 3. 2. 5. 3. 2. 4. $2\frac{1}{2}$. 5. 6. 6. -9.
7. 9. 8. 2. 9. 2. 10. $\frac{2}{3}$. 11. $\frac{4}{3}$. 12. 3.
13. $1\frac{1}{2}$. 14. $2\frac{2}{3}$. 15. $x=4,\ y=0$. 16. $x=3\frac{1}{3},\ y=5\frac{1}{10}$.
17. $x=3,\ y=2,\ z=-2$. 18. $x=\frac{1}{2},\ y=\frac{3}{2},\ z=0$.
19. $x=1,\ y=2,\ z=3$. 20. $x=3,\ y=3$.
21. $x=-4,\ y=-2$. 22. $x=1,\ y=2$.

EXERCISE 75. (Page 329.)

1. $x^3+x^2-10x+8$. 2. $x^3+5x^2-2x-24$.
3. $x^3+9x^2+26x+24$. 4. $x^3-9x^2+26x-24$.
5. $x^3-x^2(a+b+c)+x(ab+bc+ca)-abc$.
6. $x^4-6x^3-5x^2+42x+40$. 7. $x^4+4x^3-19x^2-46x+120$.
8. $x^4-x^3(a+b+c+d)+x^2(ab+ac+ad+bc+bd+cd)$
$-x(abc+abd+bcd+cda)+abcd$.
9. $x^3+x^2(p+q)-x(p^2+q^2-2pq)-(p^2-q^2)(p-q)$.
10. $x^3+x^2(y+1)-x(y^2+1-2y)-(y^2-1)(y-1)$.
11. $x^5+5x^4a+10x^3a^2+10x^2a^3+5xa^4+a^5$.
12. $x^6+6x^5a+15x^4a^2+20x^3a^3+15x^2a^4+6xa^5+a^6$.
13. $x^4-4x^3a+6x^2a^2-4xa^3+a^4$.
14. $x^5-5x^4a+10x^3a^2-10x^2a^3+5xa^4-a^5$.
15. $x^5+5x^4+10x^3+10x^2+5x+1$.
16. $a^6-6a^5b+15a^4b^2-20a^3b^3+15a^2b^4-6ab^5+b^6$.
17. $p^7+7p^6q+21p^5q^2+35p^4q^3+35p^3q^4+21p^2q^5+7pq^6+q^7$.
18. $1-6x+15x^2-20x^3+15x^4-6x^5+x^6$.
19. $32+80m+80m^2+40m^3+10m^4+m^5$.
20. $81-108a+54a^2-12a^3+a^4$.
21. $32x^5-80x^4+80x^3-40x^2+10x-1$.
22. $1-18x+135x^2-540x^3+1215x^4-1458x^5+729x^6$.
23. $1-10x+45x^2-120x^3-210x^4-252x^5+210x^6-120x^7+45x^8$
$-10x^9+x^{10}$.
24. $64+576x+2160x^2+4320x^3+4860x^4+2916x^5+729x^6$.
25. $243a^5-810a^4b+1080a^3b^2-720a^2b^3+240ab^4-32b^5$.
26. $a^5+5a^3+10a+\frac{10}{a}+\frac{5}{a^3}+\frac{1}{a^5}$.
27. $a^4+2a^3b+\frac{3}{2}a^2b^2+\frac{1}{2}ab^3+\frac{1}{16}b^4$.
28. $x^7-7x^5+21x^3-35x+\frac{35}{x}-\frac{21}{x^3}+\frac{7}{x^5}-\frac{1}{x^7}$.
29. $8x^3+8x$. 30. $2x(x^4+10x^2+5)$.

31. $4xa(3x^4+10x^2a^2+3a^4)$. 32. $864p^3q+1536pq^3$.
33. 1114. 34. 11. 35. 249. 36. 126. 37. 8.
38. 1·0253. 39. 1·218. 40. 1·0075. 41. Rs. 1·472.

EXERCISE 77. (Page 335.)

1. $\pm(2a-3b)$. 2. $\pm(4x^2+5)$. 3. $\pm(3x^2+5y^2)$
4. $\pm(9a-b^2)$. 5. $\pm(a^3-1)$. 6. $\pm(1-6x)$.
7. $\pm(x-\frac{1}{2})$. 8. $\pm(x^2y+\frac{1}{2})$. 9. $\pm(m^2-\frac{1}{3})$.
10. $\pm\left(\frac{a}{2b}-\frac{2b}{a}\right)$. 11. $\pm\left(\frac{x^2}{y^2}-\frac{y^2}{2x^2}\right)$. 12. $\pm\left(x^2-\frac{y^2}{4}\right)$.
13. $\pm\left(\frac{1}{3}a-\frac{1}{4}b\right)$. 14. $\pm\left(\frac{1}{2x}+\frac{x}{3y}\right)$. 15. $\pm(x^2-\frac{1}{2}ab)$.
16. $\pm(2x^2+3)$. 17. $\pm(3a^2+8)$. 18. $\pm\frac{x}{y}$.
19. $\pm(5b-a)$. 20. $\pm(2a^2)$. 21. $\pm\frac{4ab}{a^2-b^2}$.
22. $\pm\{3(a+b)+x+y\}$. 23. $\pm(x+\frac{1}{x}+4)$. 24. $\pm(x+\frac{1}{x}+2)$.
25. $\pm(x^2+\frac{1}{x^2}+1)$. 26. $\pm\left(x-1-\frac{1}{x}\right)$. 27. $\pm\left(x-2-\frac{1}{x}\right)$.
28. $\pm\left(x^2-2+\frac{1}{x^2}\right)$. 29. $\pm\left(x+2+\frac{1}{x}\right)$. 30. $\pm\left(x^2-2+\frac{1}{x^2}\right)$
31. $\pm(x^2-2x+1)$. 32. $\pm(2x^2+3x-5)$. 33. $\pm(x^2+xy+y^2)$.
34. $\pm(3x^2-5xy+4y^2)$. 35. $\pm\left(\frac{x}{y}+1+\frac{y}{x}\right)$.
36. $\pm\left(x+\frac{y}{4}-z\right)$. 37. $\pm\left(\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\right)$.
38. $\pm(ab+bc-ac)$. 39. $\pm(x+2y)(2x+y)(3x+y)$.
40. $\pm\frac{y+z-2x}{6}$. 41. $\pm\left(x+1+\frac{1}{x^2-1}\right)$.
42. $\pm(a^2+b^2)$. 43. $\pm(x-1)$. 46. $\pm(x^m-2x^n)$.
47. $\pm(2^{2m}+3^n)$. 48. $\pm(5x^m-2x^n)$.

EXERCISE 78. (Page 342.)

1. $4x-7y$. 2. x^2+3x+3. 3. $4x^2-3x+2$.
4. $x^2-ax+2a^2$. 5. $x^3-6x^2+12x-8$. 6. $ax^2+2x+3a$.
7. $\frac{x^2}{8}+\frac{x}{2}-1$. 8. $3x^2-3x-\frac{3}{2}$. 9. $\frac{2}{3}x^2+\frac{3}{4}x+\frac{1}{2}$.
10. $3x-4+\frac{1}{2x}$. 11. $x^2-x+\frac{1}{4}$. 12. $\frac{5}{2}-ab+3a^2b^2$.

13. $\frac{a}{2b}-1-\frac{2b}{a}$. 14. $\frac{x}{y}-\frac{1}{2}-\frac{y}{x}$. 15. $\frac{3x}{2y}-1+\frac{y}{2x}$.

16. $2x^2+3x+\frac{3}{x}$. 17. $a^3-1+\frac{1}{2a^3}$. 18. $x^{m+n}+x^{m-n}-\frac{1}{2}$.

19. $\frac{3}{2}x^{m+1}-2x^{m-1}+1$. 20. $\frac{x^2}{y^2}-2+\frac{y^2}{x^2}$.

21. $\sqrt{2}(x^2+xy+y^2)$. 22. $2x^{\frac{3}{2}}-3+4x^{-\frac{3}{4}}$. 23. $\frac{1}{2}+\frac{1}{3}x^{\frac{1}{2}}-\frac{1}{4}x^{\frac{3}{2}}$.

24. $xy^{-1}+1+x^{-1}y$. 25. $a^{\frac{2}{3}}+1+a^{-\frac{1}{3}}$. 26. $x-1$. 27. $2x-3y$.

28. $2x^2+2x+1{,}221$. 29. $x^2+x+3{,}113$. 30. $a+\frac{x^2}{2a}-\frac{x^4}{8a^3}+\frac{x^6}{16a^5}$.

31. $1+\frac{x}{2}-\frac{5}{8}x^2+\frac{5}{16}x^3$. 32. (i) $\frac{1}{4}-a$; (ii) $a-\frac{1}{4}$.

33. 2. 34. $p^2=4q$. 35. $a=12$, $b=9$.

EXERCISE 79. (Page 345.)

1. $x+4$. 2. $x-3$. 3. $2x-\frac{1}{x}$. 4. $3x-y$.

EXERCISE 80. (Page 347.)

1. $\sqrt{96}$. 2. $\sqrt[3]{500}$. 3. $\sqrt[4]{96}$. 4. $\sqrt[4]{405}$. 5. $\sqrt[5]{96}$.
6. $\sqrt[5]{972}$. 7. $\sqrt[ny]{x^{ny}}$. 8. $\sqrt[p]{x^{2p}y}$. 9. $\sqrt[5]{x^{15}y^2}$. 10. $4\sqrt{3}$.
11. $3\sqrt{7}$. 12. $2\sqrt{11}$. 13. $5\sqrt[3]{2}$. 14. $3\sqrt[4]{7}$. 15. $2\sqrt[5]{\frac{5}{2}}$.
16. $x^2\sqrt[3]{y}$. 17. $x^3\sqrt[n]{a}$. 18. $x^4\sqrt[a]{y^2}$. 19. $-8\sqrt[3]{10}$.
20. $5x^2y\sqrt[3]{2xy}$.

EXERCISE 81. (Page 349.)

1. $7\sqrt{5}$. 2. 0. 3. $4\sqrt{3}$. 4. $9\sqrt[3]{4}$. 5. $3\sqrt[3]{5}$.
6. -34. 7. $3x(2\sqrt{5x}-15\sqrt{y})$. 8. $a(x-4a)\sqrt{x}$. 9. $4a\sqrt[3]{a}$.
10. $\sqrt[12]{46656}$, $\sqrt[12]{27}$ and $\sqrt[12]{16}$. 11. $\sqrt[6]{4}$, $\sqrt[6]{16}$, $\sqrt[6]{8}$.
12. $\sqrt[12]{x^{16}}$, $\sqrt[12]{x^8}$, $\sqrt[12]{x^3}$. 13. The first is greater.
14. The second is greater. 15. The first is greater.
16. The first is greater. 17. $\sqrt[3]{4}$, $\sqrt[4]{5}$, $\sqrt{2}$.
18. $\sqrt{3}$, $\sqrt[3]{5}$, $\sqrt[4]{6}$. 19. $3\sqrt{2}$, $2\sqrt{3}$, $\sqrt[5]{7}$. 20. $\sqrt[3]{6}$, $\sqrt[4]{8}$, $\sqrt[5]{12}$.

EXERCISE 82. (Page 351.)

1. $\sqrt[3]{63}$. 2. $\sqrt[4]{60}$. 3. $\sqrt[5]{144}$. 4. $\sqrt[5]{\frac{2}{5}}$. 5. $\sqrt[6]{\frac{4}{3}}$.
6. $\sqrt[4]{\frac{4}{3}}$. 7. $\sqrt[12]{944784}$. 8. $\sqrt[5]{486000}$. 9. $\sqrt[12]{882165816}$.

10. $\sqrt[12]{\frac{1280}{27}}$. 11. $\sqrt[30]{\frac{2187}{512}}$. 12. $\sqrt[8]{32}$. 13. $18\sqrt{6}$.

14. $16\sqrt{33}$. 15. $12\sqrt[3]{4}$. 16. $35\sqrt[3]{2}$. 17. $\frac{3}{10}$.

18. $\frac{5}{3}\sqrt[6]{\frac{9}{250}}$. 19. $\sqrt[6]{432}$. 20. $2\sqrt[12]{442368}$. 21. $\sqrt[6]{360}$.

22. $\frac{1}{3}$. 23. $\sqrt[3]{\frac{7}{6}}$ 24. $x^2y\sqrt[3]{xy^2}$. 25. $288\sqrt[12]{72}$.

EXERCISE 83. (Page 351.)

1. $\sqrt{10}+\sqrt{14}+\sqrt{15}+\sqrt{21}$. 2. $\sqrt{15}-\sqrt{10}+3\sqrt{2}-2\sqrt{3}$.

3. $2\sqrt{6}-29$. 4. $12\sqrt{35}+24\sqrt{21}-12\sqrt{15}-30$.

5. $54-22\sqrt{6}$. 6. $62-20\sqrt{6}$. 7. 63. 8. $2+2\sqrt{15}$.

9. $5+3\sqrt[3]{4}+2\sqrt[3]{9}+\sqrt[3]{18}+\sqrt[3]{12}$. 10. $3-\sqrt{2}$.

11. $9\sqrt{2}+2\sqrt{3}+\sqrt{6}-6$. 12. 8. 13. $69+12\sqrt{30}$.

14. $159-24\sqrt{42}$. 15. $43-24\sqrt{3}$. 16. $2x^2-1-2x\sqrt{x^2-1}$.

17. $a^4+a^2-b^2+2a^2\sqrt{a^2-b^2}$. 18. $2a^2-2\sqrt{a^4-4b^4}$.

19. $5-4x+12\sqrt{12x-1-32x^2}$. 20. $25a^2-7b^2+24\sqrt{a^4-b^4}$.

21. $1+6\sqrt{x}+12+8\sqrt{x^3}$.

EXERCISE 84. (Page 353.)

1. $\sqrt{2}, \sqrt{5}, \sqrt{6}, \sqrt{ab}, \sqrt[3]{a^2}, \sqrt{x-1}, \sqrt{a}\sqrt[3]{b}$. 2. $\sqrt{6}$.

3. $\sqrt[3]{16}\sqrt{3}$. 4. $\sqrt{6}\sqrt[3]{36}$. 5. $\sqrt{3}-\sqrt{2}$. 6. $\sqrt{5}+\sqrt{3}$.

7. $\sqrt{a}+\sqrt{b}$. 8. $a-\sqrt{b}$. 9. $\sqrt{5}\sqrt[3]{36}\sqrt[4]{7}$.

10. $\sqrt[3]{150}.\sqrt[4]{7}$. 11. $a\sqrt{x}+b\sqrt{y}$. 12. $4\sqrt{3}-2\sqrt{5}$.

13. $3\sqrt{2}+2\sqrt{3}$. 14. $\sqrt{a+x}+\sqrt{a-x}$. 15. $x-\sqrt{x^2-1}$.

16. $\sqrt{x^2+1}+\sqrt{x^2-1}$. 17. $\frac{\sqrt{15}}{7}$. 18. $\frac{7}{5}\sqrt[3]{2}$. 19. $\frac{\sqrt[3]{20}}{2}$.

20. $\frac{\sqrt[3]{135}}{9}$. 21. ·462. 22. 1·060. 23. 1·010. 24. ·149.

25. ·671. 26. ·693. 27. 5·828. 28. −3·732.

29. 17·944. 30. 9·898. 31. 2·956. 32. 5·585.

33. 2·898. 34. 12. 35. 1154. 36. $18+13\sqrt{2}$.

37. $\frac{1}{19}(12\sqrt{6}+3\sqrt{15}+16+2\sqrt{10})$. 38. $\frac{1}{b^2}(2a^2-b^2+2a\sqrt{a^2-b^2})$.

39. $x+\sqrt{x^2-1}$. 40. $x^2+\sqrt{x^4-1}$.

53. $\sqrt{2}(1-\sqrt{2}-\sqrt{3})$ 54. $\sqrt{2}(\sqrt{3}-\sqrt{2}+1)$.

55. $\sqrt{6}(\sqrt{3}-\sqrt{2}-\sqrt{5})$. 56. $\sqrt{6}(\sqrt{2}+\sqrt{3}+\sqrt{5})$.

57. $(8-\sqrt{3})(3\sqrt{5}+2\sqrt{3}-1)$. 58. $(\sqrt{6}+2)(3\sqrt{2}+2\sqrt{3}+3\sqrt{6})$

59. $2a$. 60. $\frac{2\sqrt{a^2-b^2}}{b^2}$. 61. $2\sqrt{2}+\sqrt{3}$. 62. $\frac{\sqrt{5}+\sqrt{3}}{4}$.

63. $3\sqrt{2}-2\sqrt{3}$. 64. $4a\sqrt{a^2-1}$. 65. $3x^2$. 66. $-30\sqrt{3}$.

EXERCISE 85. (Page 360.)

1. $7+\sqrt{3}$. 2. $3-2\sqrt{2}$. 3. $6+\sqrt{5}$. 4. $4+\sqrt{3}$

5. $3-\sqrt{5}$. 6. $7+3\sqrt{5}$. 7. $5-3\sqrt{2}$. 8. $5-\sqrt{2}$

9. $3\sqrt{7}-2\sqrt{6}$. 10. $2+\sqrt{3}$. 11. $\sqrt{5}+\sqrt{3}$. 12. $\sqrt{3}+\sqrt{2}$

13. $\sqrt{7}+\sqrt{5}$. 14. $4-\sqrt{5}$. 15. $\sqrt{7}-1$.

16. $x=15$ and $y=6$. 17. $\sqrt[4]{2}\left(\sqrt{\frac{5}{2}}+\sqrt{\frac{1}{2}}\right)$.

18. $\sqrt[4]{3}(\sqrt{2}-1)$. 19. $\sqrt[4]{3}\,(\sqrt{3}+\sqrt{2})$. 20. $\sqrt[4]{2}(\sqrt{5}-\sqrt{3})$.

21. $\sqrt[4]{3}(\sqrt{3}+\sqrt{2})$. 22. $\sqrt[4]{5}\left(\sqrt{\frac{7}{2}}-\sqrt{\frac{5}{2}}\right)$.

23. $2+\sqrt{2}$. 24. $\sqrt{\frac{3}{2}}-\sqrt{\frac{1}{2}}$. 25. $\sqrt{3}-1$.

26. $\sqrt{5}-\sqrt{3}$. 27. $\sqrt{2}-1$. 28. 1. 29. $-(1+\sqrt{3})$

30. $\frac{2}{\sqrt{5}}$. 31. $\frac{5}{\sqrt{6}}$. 32. $2\sqrt{5}-\sqrt{2}+\sqrt{3}-\sqrt{6}+1$. 33. 2.

34. 1. 35. $1+\sqrt{2}+\sqrt{3}$. 36. $\sqrt{2}-\sqrt{3}+\sqrt{5}$.

37. $1+\sqrt{3}-\sqrt{5}$. 38. $2-\sqrt{5}-\sqrt{6}$.

EXERCISE 86. (Page 364.)

1. 6. 2. $\frac{5}{4}$. 3. 1. 4. 9. 5. 25. 6. $\frac{9}{20}$.

7. $\frac{3}{4}$. 8. $\frac{17}{12}$. 9. a. 10. $5\frac{3}{5}$. 11. $\frac{a}{3}$. 12. -5.

13. $-\frac{1}{2}$. 14. 3. 15. $\frac{1}{2}$. 16. $\frac{16}{25}$. 17. 5.

18. 4. 19. 5. 20. $\frac{2}{5}$. 21. 5. 22. $\frac{5}{8}$. 23. 5.

24. $-\frac{3}{28}$. 25. 8 or $-\frac{7}{3}$. 26. $-\frac{30}{11}$. 27. 5. 28. $-\frac{3}{4}$.

29. 6. 30. 3. 31. 4. 32. 8. 33. ± 5

34. ± 6. 35. -1. 36. ± 1.

37. $\pm\frac{27a^2b-(b-2a)^3}{27b}$. 38. $\pm\sqrt{\frac{(1-4b^2)^3+27b^2}{27a^2}}$.

39. $\pm\frac{2a}{\sqrt{5}}$. 40. ± 14. 41. 9. 42. 81. 43. 1.

44. $\frac{81}{a}$. 45. 5. 46. $\frac{(c^2+bc-b)^2}{a(c-1)^2}$. 47. $\frac{2}{3}$.

48. 2. 49. $\pm\frac{\sqrt{3}}{2}$. 50. -3. 51. $\frac{5}{4}$. 52. $\frac{8a}{17}$.

53. 2. 54. 0.

EXERCISE 87. (Page 370.)

1. $2m=n$. 2. $a^2m^2-b^2=0$. 3. $ab'-a'b=0$.
4. $am^2-bm+c=0$. 5. $pb^2-qab+a^2r=0$.
6. $dp^3-cqp^2+bq^2p-aq^3=0$. 7. $y^2=4ax$.
8. $2fs=v^2-u^2$. 9. $l(a-c)=m(b-d)$.
10. $lb^2-mab+na^2=0$.
11. $a''(bc'-b'c)+b''(ca'-ac')+c''(ab'-ba')=0$.
12. $2abc+ab+ac+bc=1$. 13. $ab+ac+bc+1=0$.
14. $ab+ac+bc+2abc=1$. 15. $x^2+y^2+z^2+2xyz=1$.
16. $(fy-gx)(g-f)=(x-y)^2$. 17. $(yx'-y'z)(xy'-yx')=(zx'-z'x)^2$.
18. $(c-ad)^2+b^2d=0$. 19. $(bc'-cb')(ab'-ba')^2=(ca'-ac')^3$.
20. $(bc'-cb')^2(ab'-ba')=(ca'-ac')^3$. 21. $p^3r=(r+pq)^2$.
22. $(c^2+4ab)(b^2-3ab-ac)=(4a^2-3ac+bc)^2$.
23. $a^3d^2+b^3c^2=0$ 24. $m^3q^2=n^3p^2$.
25. $4=a^2-b^2$. 26. $24=m^2-n^2$. 27. $p^2-q^2=4$.
28 $p^2-q^2=2$. 29. $q^3-p^3=3q$.
30. $p^3-q^3=3q$. 31. $q^2-p^2=2$.
32. $p^4=q^4+4q^2+2$. 33. $p^4=q^2+4q^2+2$. 34. $4b^2=a^2-b^4$.
35. $m^2+n^2=a^2+1$. 36. $a^2+b^2=r^2$.
37. $a^3+2c^3=3ab^2$. 38. $b^6+3a^2b^2=4a^3c^3$.
39. $a^6-2b^6+9a^2c^4-8a^3b^3=0$. 40. $m^3+b^2m-c^3-am^2=0$.
41. $ab+ac+bc+1=0$. 42. $a+b+c=0$. 43. $2a=b(1+c)$.
44. $p+r=4q$. 45. $m(p+q)^3=q(p+m)^3$.
46. $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. 47. $\frac{a^2}{x^2}-\frac{b^2}{y^2}=1$. 48. $x+a=0$.
49. $p^{b-c}q^{c-a}r^{a-b}=1$. 50. $a^3+b^3+c^3+abc=0$. 51. $a^2-b^2=1$.

SECTIONAL REVISION V. (Page 377.)

Paper 1. 1. (ii) (*a*) $10^5.3^2$; (*b*) $(230)^3$; (iii) (*a*) 1, (*b*) x.
2. (i) 1·0004; (ii) 1·0202. 3. (i) $81p^4+216p^3q+216p^2q^2+96pq^3+16q^4$; (ii) $81p^4-216p^3q+216p^2q^2-96pq^3+16q^4$.

4. $a=12, b=9$. 5. $7-2\sqrt{2}$. 6. $x=0, y=\frac{2}{3}, z=3$.

Paper 2. 1. (i) 300,200,000,000 *m.m.* per second,
(ii) 26,000,000,000,000 miles,
(iii) $30\times 10{,}000\times 2{\cdot}4\times 100{,}000=72\times 1{,}000{,}000{,}000$.
2. (i) 3^8 and 3^{16}; the latter is greater, (ii) 1, 1, −1, 1.
4. $12x^5+40x^3+12x$. 5. $3\sqrt{3}-2$. 6. (i) 8. (ii) 2.

Paper 3. 1. (i) $\frac{1{\cdot}6}{(10)^{24}}$, (ii) $\frac{3}{(10)^8}$, (iii) $\frac{27}{(10)^6}$. 2. (i) $\frac{1}{6}$, (ii) $\frac{1}{5}$.
(iii) 1, (iv) $\frac{5}{6}$. 3. −48. 4. $\sqrt[4]{5}(\sqrt{3}+\sqrt{2})$. 6. 1.

Paper 4. 1. (i) $\frac{1}{8}$, (ii) $\frac{1}{9}$, (iii) 81, (iv) $\frac{81}{16}$.
2. (*a*) 0, (*b*) (i) 2·301, (ii) 1·431. 3. Rs. 1·796. 4. $\sqrt{2}$.
5. (i) 40, (ii) 0. 6. $a^2+b^2+c^2-abc=4$.

Paper 5. 1. $e-\sqrt{e^2-1}$. 2. (i) $\sqrt{2}$, (ii) $9+4\sqrt{5}$. 3. $x+3-\frac{1}{x}$.
4. x^2+x-2. 5. (i) 1, (ii) $\frac{81}{a}$. 6. $(a+b+c-4)^2=abc$.

Paper 6. 1. (i) 27, (ii) 243, (iii) 16, (iv) $\frac{1}{1000}$. 2. (i) $x+x^{\frac{1}{2}}y^{\frac{1}{2}}+y$,
(ii) $x^{\frac{3}{4}}+x^{\frac{1}{2}}y^{\frac{1}{4}}+x^{\frac{1}{4}}y^{\frac{1}{2}}+y^{\frac{3}{4}}$. 3. $\frac{1}{2}x+\frac{1}{3}y+3$. 4. (i) 3, (ii) 1.
5. (i) $\frac{4\sqrt{5}+3\sqrt{15}-10\sqrt{3}-6}{11}$, (ii) 0. 6. $x^m=nx$.

Paper 7. 1. (i) $a^{\frac{2}{3}}-a^{\frac{1}{3}}b^{\frac{1}{3}}+b^{\frac{2}{3}}$, (ii) $x-y$. 2. (i) $1+6x+15x^2+20x^3+15x^4+6x^5+x^6$, (ii) $1-6x+15x^2-20x^3+15x^4-6x^5+x^6$.
3. $x^2+\frac{x}{2}+\frac{1}{x}$. 4. (i) 2, (ii) $3\frac{3}{4}$. 5. (i) ·414, (ii) $\frac{1}{x^2nr}$. 6. $a+3b=b^3$.

Paper 8. 1. 1·4. 2. $x=-\frac{3}{2}, y=-2$. 3. (i) $x^5+5x^4a+10x^3a^2+10x^2a^3+5xa^4+a^5$,
(ii) $x^5-5x^4a+10x^3a^2-10x^2a^3+5xa^4-a^5$.
4. $2x^2-3x+5$. 5. $\frac{2n-1}{2n}$. 6. $a^2b^2(a^2+b^2+3)=1$.

EXERCISE 88. (Page 383.)

1. ± 4. 2. ± 2. 3. 10 or 0. 4. ± 2. 5. $\pm\sqrt{5}$.
6. $\pm\frac{3}{\sqrt{2}}$. 7. $\pm a$. 8. ± 2. 9. ± 1. 10. ± 3. 11. ± 2.
12. ± 2. 13. ± 9. 14. $\pm\frac{1}{\sqrt{11}}$. 15. $\pm\frac{1}{\sqrt{2}}$.
16. $\pm\sqrt{a^2-2a}$. 17. 4, −8.

EXERCISE 89. (Page 384.)

1. 4, 6. 2. $-4, -6$. 3. $5, -7$. 4. $-8, 9$. 5. 1, 3.

6. $2, -5$. 7. $a, -b$. 8. $\frac{3}{2}, \frac{4}{3}$. 9. $\frac{1}{6}, -\frac{2}{5}$.

10. $\frac{b}{a}, -\frac{c}{b}$. 11. $-6, \frac{2}{9}$. 12. $0, -5$. 13. -4.

14. $\frac{7}{3}$. 15. $-\frac{a}{3}$. 16. $\frac{b}{a}$. 17. $p+q$.

18. $p+q, p-q$. 19. 7, 2. 20. $-6, -4$. 21. $4, -3$.

22. $-\frac{1}{4}, -\frac{1}{3}$. 23. $-9, -3$. 24. $\frac{1}{5}, -1$.

25. $\frac{2}{3}, \frac{3}{2}$. 26. $3, \frac{1}{3}$. 27. 9, -4. 28. $10, -\frac{1}{16}$.

29. $\frac{3}{2}, \frac{1}{4}$. 30. $\frac{10}{3}, -\frac{2}{5}$. 31. $x^2-4x-21=0$.

32. $x^2-16=0$. 33. $x^2+x(b-a)-ab=0$.

34. $x^2-3x-180=0$. 35. $x^2-2mx+m^2-n^2=0$.

EXERCISE 90. (Page 387.)

1. 9, -5. 2. 11 or -5. 3. $\frac{5}{3}, -6$. 4. $\frac{3}{2}, \frac{2}{3}$.

5. $-\frac{1}{2}, \frac{9}{4}$. 6. $\frac{4}{3}, -\frac{3}{5}$. 7. $-\frac{3}{5}, 1$. 8. $\frac{25}{2}, -\frac{1}{2}$.

9. 9, 8. 10. $\frac{b \pm \sqrt{b^2+4ac}}{2a}$.

EXERCISE 91. (Page 388.)

1. $2, -\frac{1}{3}$. 2. $1, -\frac{1}{3}$. 3. $2, \frac{1}{3}$. 4. $\frac{3}{4}, \frac{2}{3}$.

5. 14, $\frac{9}{4}$. 6. $-\frac{3}{2}, \frac{2}{5}$. 7. $2, \frac{20}{3}$. 8. $-2, -\frac{3}{7}$.

9. $\frac{3}{5}, 4$. 10. $\frac{2p}{3}, -\frac{4p}{5}$. 11. $2, -p$. 12. $\frac{p}{q}, -\frac{q}{p}$.

EXERCISE 92. (Page 390.)

1. 0 and $\frac{1}{3}$. 2. $\frac{4}{3}$ or $\frac{3}{4}$. 3. $\pm 13, \pm 14$.

4. 12, 14 or $-12, -10$. 5. $\pm 11, \pm 13$. 6. $\pm 5, \pm 12$.

7. 17 ft., 15 ft. 8. (i) $AP=a(\pm\sqrt{3}-1)$, (ii) $AP=a(2\pm\sqrt{3})$.

(ii) $AP=a$ or $\frac{3a}{4}$, (iv) $AP=\frac{a(2\pm\sqrt{2})}{4}$. 9. 24 ft., 10 ft.

10. 30 ft., 16 ft. 11. 50 feet, 18 feet. 12. 11 hours.
13. 15 miles per hour. 14. 4 yards. 15. 10 ft., 24 ft., 26 ft.
16. 6 hours 40 minutes. 17. 12 feet, 8 feet. 18. 6 feet and 9 feet.
19. 24 ft., 18 ft., 15 ft. 20. 1024 men.

EXERCISE 93. (Page 398.)

1. (i) 1·96, (ii) 5·29, (iii) 10·24, (iv) 1·73, (v) 2·70, (vi) 3·67. 5. (i) 2, −3, (ii) 1, 3; (iii) $-3, -\frac{1}{2}$. 6. (i) 3, 2; (ii) $\frac{1}{2}, -3$; (iii) $\frac{1}{2}, -\frac{3}{2}$. 7. $\frac{1}{2}, -3$. 8. $\frac{2}{3}, -2$ 9. $4, -\frac{2}{3}$.
10. 2, 144.

EXERCISE 94. (Page 403.)

1. -1. 2. -37. 3. $4a^3-2a^2+5a-3$.
4. $-(5m^3+m^2+3m+4)$. 5. $-56a^4$.
11. $(x-1)(x^2+2x+3)$. 12. $(x-1)(x-2)(x-3)$.
13. $(x+1)(x^2+8x+15)$. 14. $(x-1)(3x^2+4x+5)$.
15. $(x+1)(4x^2-x+2)$. 16. $(x-1)(x+1)(2x^2+3x+4)$.
17. $(x-1)^2(x^2+4x+2)$. 18. $(x+1)^2(x^2-3x+5)$.
19. $(x-1)(x+1)(x+3)(x-2)$. 20. $(x-1)^2(2x^2+3x-4)$.
21. 3. 22. $a+b+c+d=0$. 23. $a=-2, b=5$.
24. -600. 26. $\frac{3}{2}$.
27. $(x-3)(x+2)(x+4)$.
28. $(x+2)(x-3)(x-4)$. 29. $(x+2)(3x-1)(2x-3)$.
30. $(x-5)(x+2)(x+3)$. 31. $(x-2)(x-5)(x+7)$.
32. $(x+2)(x^2-3x+6)$. 33. $(x+3)(x^2-3x+4)$.
34. $(x-4)(x-3)(x^2+5x+3)$.

EXERCISE 95. (Page 409.)

8. $4a^2+2ax+x^2$. 9. $a^2-4ab+16b^2$.
10. $32a^5-16a^4x+8a^3x^2-4a^2x^3+2ax^4-x^5$.
11. $9m^2+3m+1$. 12. $x^3+4x^2y+16xy^2+64y^3$.
13. $1-2m+4m^2-8m^3$. 14. $16a^4-8a^3b+4a^2b^2-2ab^3+b^4$.

EXERCISE 96. (Page 414.)

1. $a_0x^8+a_1x^7+a_2x^6+a_3x^5+a_4x^4+a_5x^3+a_6x^2+a_7x+a_8$.
2. $a_0x^4+a_1x^3y+a_2x^2y^2+a_3xy^3+a_4y^4+a_5x^3+a_6x^2y+a_7xy^2+a_8y^3+a_9x^2+a_{10}xy+a_{11}y^2+a_{12}x+a_{13}y+a_{14}$.
3. Fractional and irrational. 4. Fractional and rational.

5. Integral and irrational. 6. Fractional and rational.
7. Integral and rational. 8. Fractional and irrational.
9. (i) 4, (ii) 9, (iii) 12, (iv) 5, (v) $6x^2-8x+10$, (vi) $9k^2+12k+15$, (vii) $3m^2+2m+4$. 10. (i) 14, (ii) 6, (iii) $2(1-p^2)$. 11. $2(b+2ax)$. 12. 7.
13. $2x^4-5x^3-2x^2+11x-6$. 14. $x^5-5x^4+11x^3-8x^2-5x+12$.
15. $2x^5-10x^4+3x^3+16x^2-14x+3$.
16. $14x^7-32x^5-21x^4+14x^3+6x^2-12x-9$.
17. $15x^6+5x^5-6x^4-x^2-2x-1$.
18. $5x^7-13x^6+9x^5-6x^3+4x^2-3x-1$.
19. $12x^8+2x^7-18x^6-3x^5+22x^4+4x^3+3x^2-4$.
20. $\frac{2}{5}x^4-\frac{13}{7}\frac{6}{5}x^3y+\frac{8}{5}x^2y^2+\frac{3}{10}xy^3-y^4$.
21. $\cdot 2x^4+\cdot 07x^3y+\cdot 04x^2y^2+\cdot 19xy^3-\cdot 1y^4$.
22. $2x^3+5x^7-2x^6-9x^5-x^4-5x^3-14x^2-3x+3$.
23. $6x^3-x^2(5+6a)+x(5a-6)+6a$, $-\frac{5}{6}$.
24. x^2-x. 25. $10x^2-3x-12$. 26. x^3+2x^2-3x+1.
27. $2x-1$. 28. x^3-x^2+x-1. 29. x^4-x^2+1.

EXERCISE 97. (Page 422.)

1. Homogeneous and symmetrical, absolute symmetry.
2. Homogeneous, but not symmetrical.
3. Neither homogeneous nor symmetrical.
4. Homogeneous, but not symmetrical.
5. Homogeneous, but not symmetrical.
6. Homogeneous, but not symmetrical.
7. Homogeneous and symmetrical, absolute symmetry.
8. Symmetrical, but not homogeneous, absolute symmetry.
9. Homogeneous, but not symmetrical.
10. Homogeneous and symmetrical, absolute symmetry.
11. Not homogeneous but symmetrical, absolute symmetry.
12. Homogeneous and symmetrical, cyclic symmetry.
13. ax^5, bx^4y, cx^3y^2. 14. ax^4, bx^3y, cx^2y^2, dx^2yz.
15. $a(x^4+y^4+z^4)+b(x^3y+x^3z+y^3z+y^3x+z^3x+z^3y)+c(x^2y^2+x^2z^2+y^2z^2)+d(x^2yz+y^2zx+z^2xy)$. 16. $2x^2+3xy+2y^2$.
17. $2(x^3+y^3)+3(x^2y+xy^2)$. 18. $(x+a)(x+b)+(x+a)(x+c)+(x+b)(x+c)$.
19. $(x-a)(b-a)(c-a)+(x-b)(c-b)(a-b)+(x-c)(a-c)(b-c)$.
20. $(x+a)(x+b)(x-c)+(x+b)(x+c)(x-a)-(x+c)(x+a)(x-b)$.
21. $\frac{x^2(a+b)}{ab}+\frac{x^2(b+c)}{bc}+\frac{x^2(c+a)}{ca}$.

ANSWERS

22. $\dfrac{bc}{(a-b)(a-c)}+\dfrac{ca}{(b-c)(b-a)}+\dfrac{ab}{(c-a)(c-b)}$.

23. $\dfrac{1}{(a-b)(a-c)(x-a)}+\dfrac{1}{(b-c)(b-a)(x-b)}+\dfrac{1}{(c-a)(c-b)(x-c)}$.

24. $3x^2+2x(a+b+c)+(ab+ac+bc)$.

25. $2(a^2+b^2+c^2-ab-ac-bc)$.

26. $3x^2+2x(a+b+c)$.

27. $ab(a-b)+bc(b-c)+ca(c-a)$.

28. $\Sigma x^2+3\Sigma xy$.

29. $\Sigma x^2-\Sigma xy$.

30. $x^3y+y^3z+z^3x-x^2y^2-y^2z^2-z^2x^2$.

31. $3\Sigma a^2-2\Sigma ab$.

32. (i) $a=b=7$, and $c=d=-3$, (ii) $p=4$, and $q=-2$.
(iii) $m=-n=2$, and $q=-p=7$.

EXERCISE 98. (Page 427.)

1. $a=2$, $b=8$ and $c=-3$.
2. $a=3$, $b=-9$, $c=0$, $d=15$.
3. $l=1$, $m=1$, $n=-11$, $k=-23$.
4. $a=2$, $b=0$, $c=12$, $d=0$, $e=2$.
5. $a_0=2$, $a_1=2$, $a_2=-6$, $a_3=-10$, $a_4=-14$.
6. 18.
7. $a=2$, $k=-8$.
8. $A=1$, $B=2$.
9. $A=2$, $B=-3$.
10. $A=1$, $B=8$, $C=3$.
11. $A=0$, $B=2$, $C=3$.
12. $A=1$, $B=-1$, $C=0$, $D=2$.
13. $3(x-1)^2+2(x+2)+1$.
14. $3(x+1)^3-2(x+1)+1$.
15. $(x+1)^3-4(x+1)^2+(x+1)-4$.
16. $A=1$, $B=1$, $C=0$.
17. $3\Sigma x^2y+6xyz$.
18. $24xyz$.
19. 0.
20. $3x^2-x+2$.
21. $3x^3-2x^2+4$.
22. $2x^3-3x+5$.
23. $a=1$, $b=-2$.
24. $p=\pm30$, $q=19$ or 31.
25. a^2-a+2.
26. x^2-3x-2.
27. $2x^2+x-3$.
28. x^2-2x+3.
29. $p=-3$.
30. $a=7$, $b=1$.
31. $\dfrac{-1}{(x+1)}-\dfrac{1}{5(x-3)}+\dfrac{6}{5(x+2)}$.
32. $\dfrac{3}{2x+1}+\dfrac{7-6x}{4x^2-2x+1}$.
33. $\dfrac{1}{(x-2)}-\dfrac{x+4}{(x+1)^2}$.
34. $A=3$, $B=-1$, $C=2$.
35. $A=1$, $B=2$, $C=D=-1$.

EXERCISE 99. (Page 432.)

1. $(x-1)(x^2+2x+3)$.
2. $(x-1)(x-2)(x-3)$.
3. $(x-1)(x-3)(x+5)$.
4. $(x-3)(x+2)(x+4)$.
5. $(x+2)(x+3)(x+5)$.
6. $(x-1)(x+1)(x+5)$.
7. $(x-5)(x+2)(x+3)$.
8. $(x+2)(x^2-3x+6)$.
9. $(x-1)(x+1)(x^2-2x-2)$.
10. $(x-4)(x-3)(x^2+5x+3)$.
15. $(a-b)(b-c)(c-a)$.
16. $3(a-b)(b-c)(c-a)$.
17. $(a+b)(b+c)(c+a)$.
18. $(a+b)(b+c)(c+a)$.

19. $24abc$.
20. $-(a-b)(b-c)(c-a)(a+b+c)$.
21. $(a-b)(b-c)(c-a)(a+b+c)$.
22. $(a+b+c)(ab+ac+bc)$.
23. $2(a-b)(b-c)(c-a)(a+b+c)$.
24. $5(x-y)(y-z)(z-x)(x^2+y^2+z^2-xy-yz-zx)$.
25. $(a-b)(b-c)(a-c)(a^2+b^2+c^2+ab+bc+ca)$.
26. $5xy(x+y)(x^2+xy+y^2)$.
27. $5(x+y)(y+z)(z+x)(x^2+y^2+z^2+xy+yz+zx)$.
28. $-(x+y)(y+z)(z+x)(x-y)(y-z)(z-x)$.
29. $12xyz(x+y+z)$.

EXERCISE 101. (Page 438.)

1. $(a-b)(b-c)(a-c)$.
2. $(a+b)(b+c)(c+a)$.
3. $(a-b)(b-c)(a-c)(a^2+b^2+c^2+ab+bc+ca)$.
4. $-(a-b)(b-c)(c-a)(ab+bc+ca)$.
5. $-(a+b)(b+c)(c+a)(a-b)(b-c)(c-a)$.
6. $(a-b)(b-c)(a-c)(a^3+b^3+c^3+a^2b+b^2a+a^2c+ac^2+b^2c+bc^2+abc)$
7. $(3x-5y)(6x-7y)$.
8. $(3a-5b)(9a+4b)$.
9. $(5x-3y)(11x+4y)$.
10. $x^2y^2(8x+7y)(15x-4y)$.
11. $(4x-5y)(11x-4y)$.
12. $(3x-7y)(7x-3y)$.
13. $(4x+y)(7x-20y)$.
14. $(a+x-2)(a+y+2)$.
15. $(x+1)(ax-x-a)$.
16. $(x+a+2)(x-a-1)$.
17. $(x-a)(x-b-1)$.
18. $(bx-x+1)(bx+x+b)$.
19. $(a-b-3)(a-b+2)$.
20. $(x+b+1)(x+b-1)$.
21. $(x+a)(x+b+c)$.
22. $(x-1)(x-3)(x+4)$.
23. $(x-4)^2(x+2)$.
24. $(x-3)(x^2+3x+6)$.
25. $(x+3)(x^2-3x+4)$.
26. $(x+2y)(x^2-5xy+10y^2)$.
27. $(2x-1)(4x^2+2x+3)$.
28. $(a-b)(2a^2+ab+b^2)$.
29. $(3x-1)(9x^2+3x+5)$.
30. $(x^2-3x-1)(x^2+2x-1)$.
31. $(x-1)^2(x^2+6x+1)$.
32. $(x^2-4x+1)(x^2-x+1)$.
33. $(x^2+x+1)(2x^2+x+2)$.
34. $(x^2-xy-y^2)(3x^2+2xy-3y^2)$.
35. $(2x^2-xy+2y^2)(x+2y)(2x+y)$.
36. $(x^2-xy-y^2)(6x^2-19xy-6y^2)$.
37. $(a-2b+3c)(2a+3b-5c)$.
38. $(x+2y-3z)(2x-y+z)$.
39. $(x+y+z)(x-2y+3z)$.
40. $(a-x+2y)(a+2x-y)$.
41. $(x+2a+b)(x-a+2b)$.
42. $(x+y+3z)(x-2y+z)$.
43. $(2x-y+z)(x+3y-2z)$.
44. $3(2a-b)(b-3c)(3c-2a)$.
45. $3(a-2b+1)(a+b-1)(b-2a)$.
46. $3(a-1)(a-2)(3-2a)$.
47. $6a(a+b+1)(b+1-a)$.
48. $3(2x+y)(x-y)(x+2y)$.
49. $3abc(a-b)(b-c)(c-a)$.

ANSWERS

50. $-5(x-y)(y-z)(z-x)\{(x-y)(y-z)+(y-z)(z-x)+(z-x)(x-y)\}$.

51. $5(x-1)(x-2)(3-2x)(3x^2-9x+7)$.

52. $(a+b+c)(ab+bc+ca)$.

53. $-(a+b+c)(a^2+b^2+c^2-2ab-2ac-2bc)$.

54. $(a+b+c)(a^2+b^2+c^2)$.

55. $(a+b+c)(b+c-a)(c+a-b)(a+b-c)$.

56. $(a+b+c)(ab+bc+ca)$.

57. $(x^2-2x+4)(x^2-3x+4)$.

58. $(x^2+4x+10)(x^2+4x-2)$.

EXERCISE 102. (Page 445.)

1. 0. 2. 1. 3. 1. 4. $\frac{1}{abc}$. 5. 0. 6. $\frac{1}{abc}$.

7. 0. 8. 0. 9. 2. 10. $ab+bc+ca$.

11. $a^2+b^2+c^2+ab+bc+ca$. 12. $a+b+c$. 13. 0.

14. $\frac{1}{abc}$. 15. $a+b+c$. 16. 1. 17. $-\frac{(a-b)(b-c)(c-a)}{(a+b)(b+c)(c+a)}$.

18. 1. 19. 2. 20. -3. 21. x^2.

22. $\frac{1}{(x-a)(x-b)(x-c)}$. 23. $\frac{x^2}{(x-a)(x-b)(x-c)}$.

24. $\frac{1}{(x+a)(x+b)(x+c)}$. 25. $\frac{x^2}{(x+a)(x+b)(x+c)}$.

26. $\frac{x+1}{(x-a)(x-b)(x-c)}$. 27. $\frac{1+px+qx^2}{(x-a)(x-b)(x-c)}$.

28. $(a+b)(b+c)(c+a)$. 29. $ab+bc+ca$. 30. 3.

31. $\frac{3}{2}(a+b+c+d)$. 32. -1. 33. $x+y+z$.

EXERCISE 103. (Page 450.)

1. $\frac{1}{b}$. 2. $\frac{1}{m}$. 3. $\frac{a^4-10a^3b-6ab^3-b^4}{a^4+10a^3b+6ab^3-b^4}$.

4. 0. 5. $2a^2(2a^2-1)$. 6. 2. 7. 0.

EXERCISE 105. (Page 456.)

1. 2. 2. 3. 3. x^2+2x+3. 4. x^2-x-12, x^2-6x+8.

7. $(ac'-ca')^3=(cb'-bc')(ab'-ba')^2$. 8. 8. 9. $p=0$.

10. $a=4$, H.C.F. $=x+1$, $a=-14$, H.C.F. $=x-2$.

SECTIONAL REVISION VI. (Page 457.)

Paper 1. 1. $a_0x^3+a_1x^2y+a_2xy^2+a_3y^3+a_4x^2+a_5xy+a_6y^2+a_7x+a_8y+a_9$. 2. (i) $15x^7-x^5+13x^4-14x^3-3x^2-2x-8$. (ii) 3. $p=q=9$ and $a=b=-4$. 4. $A=-1$, $B=2$. 5. 6. 6. $(7x+11y)(8x-21y)$. . 7. $x^2-x-20=0$.

Paper 2. 1. (i) $-1, 1, -13$, (ii) $x+1$.

2. $a_0(x^4+y^4+z^4)+a_1(x^3y+x^3z+y^3x+y^3z+z^3x+z^3y)+a_2(x^2yz+y^2xz+z^2yx)$

3. $\dfrac{x^2(a-b)}{ab}+\dfrac{x^2(b-c)}{bc}+\dfrac{x^2(c-a)}{ca}$.

4. $a_0=1$, $a_1=-1$, $a_2=0$, $a_3=5$, $a_4=-3$.

5. $5(2x-y)(2y-z)(z-y-2x)(4x^2+3y^2+z^2-2xz-3yz)$.

6. $\dfrac{x^2}{(x-a)(x-b)(x-c)}$. 7. $x^2-x-56=0$.

Paper 3. 3. $\dfrac{x^2-y^2}{x(y-z)}+\dfrac{y^2-z^2}{y(z-x)}+\dfrac{z^2-x^2}{z(x-y)}$.

4. $\dfrac{15}{4(x-1)}+\dfrac{1}{20(x+3)}-\dfrac{14}{5(x-2)}$.

5. $3(2x-3y+z)(x+2y-z)(y-3x)$. 6. 11, -5.

Paper 4. 1. $-\{a^3(b-c)+b^3(c-a)+c^3(a-b)\}$.

2. $5x^2+3x-1+\dfrac{7}{x-1}-\dfrac{16}{x-2}$.

3. $\frac{1}{2}(x^2+y^2+z^2)+\frac{1}{3}(xy+yz+zx)$. 5. $(x-3)(x+4)(x-1)$.

6. $x=3$, $y=5$, $z=1$. 7. $\frac{5}{2}$, 3.

Paper 5. 1. $A=-2$. $B=3$, $p=2$. 3. $4(a^2+b^2+c^2-ab-bc-ca)$.

5. $(10x^2-23x-10)(x^2-4x-1)$. 6. $\frac{7}{6}$, $-\frac{5}{2}$.

7. x^3-3x^2+4x-1.

Paper 6. 1. (i) $a=-9$, $b=4$; (ii) $a=1$, $b=-4$.

4. $\dfrac{ab(a+b)+bc(b+c)+ca(c+a)}{abc(a+b)(b+c)(c+a)}$. 6. $(2x+1)(4x^2-2x+3)$.

7. $-\frac{7}{3}$, $\frac{3}{5}$.

Paper 7. 2. (i) $5ab(a+b)(a^2+ab+b^2)$;

(ii) $(a-b)(b-c)(c-a)(ab+bc+ca)$. 3. $-\dfrac{1}{abc}$. 5. $x=-3$.

6. $3x^2+8xy+4y^2$, $2x^2+xy-6y^2$. 7. $\frac{2}{3}$, 5.

Paper 8. 1. (i) $(x+2a-3)(x-a+2)$, (ii) $(a-2c+b)(2a+c-3b)$.

3. $p=1$, 2 or 5.

4. $x=\dfrac{1}{(a-c)(a-b)}$, $y=\dfrac{1}{(b-a)(b-c)}$, $z=\dfrac{1}{(c-a)(c-b)}$.

5. 1. 6. $\dfrac{3}{x-1}+\dfrac{2(2x+1)}{x^2+x+1}$. 7. 5, $\frac{3}{2}$.

MISCELLANEOUS EXERCISES. (Page 461.)

1. (*a*) $(5370)^3$, (*b*) 180. 2. (*a*) $6x^7-19x^6+10x^5+5x^4-15x^3+20x^2+x-4$, (*b*) 24. 3. x^2+x+2. 5. (i) $(a-b)(a-ab+b)$, (ii) $(a+b+c)(2ab+2bc+2ca-a^2-b^2-c^2)$, (iii) $(a+b)(a-b)\left(1+\dfrac{1}{ab}\right)\left(1-\dfrac{1}{ab}\right)$. 6. 1.

7. (i) $x=5, y=3, z=7$, (ii) $a+b+c$.

9. $\dfrac{5}{x+2}-\dfrac{2}{x+3}$. 10. Length = 21 ft., breadth = 15 ft.

11. (*a*) $(\cdot034)^3$, (*b*) $\frac{1}{2}$. 12. (*a*) $2x^3+5x^2-3x+1$, (*b*) 4.

13. x^2+x-4. 15. (i) $(a+1)(a-1)(b+1)(b-1)$, (ii) $(y-x)(xy+a)$, (iii) $(18x^2+6x+1)(18x^2-6x+1)$, 1861×1741.

16. x^2. 17. (i) 5, (ii) 5. 18. $abc=1$.

19. $\dfrac{3}{x-2}-\dfrac{3x+1}{(x+1)^2}$. 20. (i) 46 ft., (ii) Length 32 ft., breadth 14 ft.

21. (i) 16,27,81, (ii) $2a+2\sqrt{a^2-b^2}$. 23. x^2+2x+3.

25. (i) $(x-37)(x+36)$, (ii) $(x^2-yz)(z-y)$. 26. $5+\sqrt{6}$.

27. (i) $\frac{79}{29}$, (ii) -2. 28. $x^4-y^4=4x^2$. 29. $A=\frac{1}{2}$, $B=-4$, $C=4\frac{1}{2}$. 30. 24 men, Rs. 27. 31. (*a*) 27, $\frac{1}{216}$. 32. (*b*) $\dfrac{5}{x}$.

33. $2x^2+3x+2$. 35. (i) $(ax-a+1)(ax+x+a)$, (ii) $(x^2+y^2-1)^2$.

36. $\dfrac{x^2+x-3}{x^2-2}$. 37. (i) $x=3, y=2$, (ii) $x=2, y=-4, z=2$.

39. $A=6, B=5, C=-6, D=-4$. 40. Rs. 2725.

42. (*a*) $8x^3$, (*b*) 8. 43. x^2-2x+1.

45. (i) $(x+4)(x^2-4x+16)$, (ii) $(x+y)(y+z)(z+x)$. 46. $\frac{1}{3}$.

47. (i) 4, (ii) $x=y=2$. 48. $abc=1$. 49. $A=4$, $B=-6$, $C=-3$.

50. 30 miles. 52. (i) 33, (ii) $\dfrac{4x+3y}{29}=\dfrac{3x-2y}{9}$.

53. $x^2+2-\dfrac{1}{x}$. 54. $x-\sqrt{x^2-1}$. 55. $(x^2+4x+8)(x^2-4x+8)$, (ii) $(x+a)(x+b+c)$. 56. $\dfrac{1}{(x-a)(x-b)(x-c)}$.

57. (i) 2, (iii) $\left(\dfrac{2}{m}\right)^{\frac{1}{2-m}}$. 58. $\frac{7}{12}$, $4\frac{8}{15}$. 59. $p=6, q=11, r=6$.

60. 72 sheep of each kind. **61.** (*a*) $\left(\frac{2645}{9}\right)^3$, (*b*) $\frac{1}{25}$.

62. (*a*) 83, (*b*) 21.

63. x^2+x-6, x^2+2x-3. **65.** (i) $(2x^2+1)(2x^2-1)(2x^2+2x+1)$ $(2x^2-x+1)$, (ii) $6(x-y)(5x+3y)(3x+5y)$. **66.** $\frac{x(x+1}{x-1}$

67. (i) $x=5, y=2$, (ii) $\frac{4}{3}$. **68.** k^3+3k.

69. $k=3, l=0; m=-1$. **70.** 63. **71.** 2.

72. (*a*) -1, (*b*) (i) $4-x$, (ii) $x-4$, (iii) 4.

73. x^3-7x-6. **76.** 1. **77.** (i) 5 or $\frac{5}{4}$, (ii) $x=4\frac{1}{2}, y=1\frac{1}{2}$.

79. $-1, 0, 6x^2-12x+8$. **80.** $32\frac{4}{13}$ minutes past 5.

81. (*a*) $\frac{1}{\sqrt{a^2+b^2}}$, (*b*) $\frac{x^2-y^2}{\sqrt{a^2-4b^2}}$. **82.** (*a*) 0, (*b*) x^6+x^4+1.

83. $\pm\sqrt{3}$. **84.** $a^4-4a^2b+2b^2$. **85.** $\frac{x}{x-3}$. **86.** 0.

88. $5x^2+2xy+5y^2$. **89.** $12abc(a+b+c)$. **90.** 80 men

91. (*a*) 1. **92.** $1+x+x^2+x^3+x^4+x^5+x^6+x^7$.

93. (*b*) $(x-a)(x+b)(x+c)$
$\equiv x^3+x^2(-a+b+c)+x(-ab-ac+bc)-abc$.

94. $(3a+2b)(2a-3b)(a-b)(4a^2+6ab+9b^2)$.

95. (*a*) $(a-2b)(a+5b)(a-3b)$, (*b*) $(x-2)(x^2+2x-13)$.

96. 2. **97.** $3\sqrt{2}-2$. **100.** 600 men.

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